Vibration waveform effects on dynamic stabilization of ablative Rayleigh-Taylor instability

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An analysis of dynamic stabilization of Rayleigh-taylor instability in an ablation front is performed by considering a general square wave for modulating the vertical acceleration of the front. Such a kind of modulation allows for clarifying the role of thermal conduction in the mechanism of dynamic stabilization. In addition, the study of the effect of different modulations by varying the duration and amplitude of the square wave in each half-period provides insight on the optimum performance of dynamic stabilization.

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I. INTRODUCTION

A considerable effort in inertial confinement fusion (ICF) research, including fast and shock ignition scenarios, is being done for finding methods that could potentially reduce the driver energy required for ignition and high gain.\textsuperscript{1-4} Such a reduction comes essentially from the fact that those approaches to ICF are expected to be less sensitive to the deleterious effects of the Rayleigh-Taylor instability (RTI) than the central ignition concept. In fact, any method for controlling the instability growth during the implosion process has the potential for reducing the driver energy. In this framework, dynamic stabilization driven by the vertical vibration of the front has been proposed for the reduction of the growth rate and of the minimum perturbation wave numbers $k = 2\pi/\lambda$ ($\lambda$ is the perturbation wavelength) that is stable.\textsuperscript{5,6}

Dynamic stabilization of RTI has already been demonstrated in Newtonian fluids by forcing a sinusoidal oscillation of the interface separating the two fluids,\textsuperscript{7-9} and the analysis of those experiments have shown that viscosity and surface tension are both necessary and play a crucial role in making possible the stabilization of all the wave numbers larger than a certain minimum value $k_m$.\textsuperscript{10,11} These experiments are of relevance for applications to ICF because of the close analogies existing between the viscosity and surface tension in Newtonian fluids, and the effects of the mass ablation rate on the evolution of the RTI in an ablation front.\textsuperscript{12}

Most of the former studies on dynamic stabilization of RTI have considered a sinusoidal driving of the interface requiring rather complicated numerical calculations in order to find the region of stability. Such an approach makes very difficult to obtain the similarity relationships that rules the problem and that are necessary for the design and interpretation of experiments. Recently, we have shown that the essence of the problem can be captured by using the simplest possible driving waveform which consists in a sequence of Dirac deltas.\textsuperscript{11,13} In such a way, we have been able to develop a completely analytical treatment that yields explicit expressions for the stability regions and for the instability growth rates as functions of the parameters of the acceleration modulation boundaries as well as of the parameters of the steady ablation front. As in the case of Newtonian fluids, we have found that the damping effect equivalent to the viscosity produced by the ablation process by itself is crucial for the dynamic stabilization of the front. Instead, the effect equivalent to the surface
tension that in the ablative RTI appears when the front is at least partially driven by thermal conduction, resulted to be beneficial but not essential for the dynamic stabilization.\textsuperscript{13} This result was of special relevance for the scenario of ICF ablatively driven by ion beams recently considered by Logan et al.\textsuperscript{14} in which the driver energy is transported up to the front mostly by classical Coulombian collisions. In such a case, the absence of transport by thermal conduction prevents the surface tension-like effect that stabilize the wave numbers larger than the cut-off value \( k_c \).\textsuperscript{12}

In this paper we show that such a different behavior of the dynamic stabilization in ablative RTI in comparison with RTI in Newtonian fluids is not general but rather specific of acceleration modulation involving Dirac deltas. We consider a more general type of driving for the dynamic stabilization of ablative RTI consisting in a two-step or square wave modulation of the front acceleration which should be more representative of experimentally accessible drivings.\textsuperscript{15} This type of waveform can still be treated analytically and has also the advantage that allows for considering asymmetries in the acceleration amplitudes and in the duration of each half-period. In this manner we can analyze the effect of different modulations and get some insight about the optimization process of dynamic stabilization of ablative RTI.

II. THE BASIC EQUATIONS OF THE MODEL

We consider the problem of RTI instability of an ablation front in the framework of the sharp boundary model as in Refs.[16-21]. In this model the front is taken as a zero thickness interface placed at \( y = \xi(x,t) \) that separates two fluids of densities \( \rho_2 \) and \( \rho_1 < \rho_2 \), ahead and behind the front, respectively. The gravitational field \( \vec{G}(t) \) is pointing in the positive \( y \)-axis direction which is taken in the direction opposite to the density gradient. In the sharp boundary model for the ablation front, the information regarding the density profiles at both sides of the front is incorporated by considering the self-consistent density jump \( r_D = \rho_1/\rho_2 \), with \( \rho_1 \) and \( \rho_2 \) taken as the densities at a distance \( k^{-1} \) of the interface: \( \rho_1 = \rho(y = k^{-1}) \) and \( \rho_2 = \rho(y = -k^{-1}) \approx \rho(y = 0) \). With these assumptions we can get the equation of motion of the interface due to the instability as in Refs.[12,16, 20, 22-25]:

\[
\ddot{\xi} + \frac{4k\nu_2}{1 + r_D} \dot{\xi} + \left[ \phi_0 \frac{k^2\nu_2^2}{r_D} - A_T kG(t) \right] \xi = 0 ,
\] (1)
FIG. 1. General square wave modulation $\Gamma(\tau)$ of the acceleration that drives the dynamic stabilization of Rayleigh-Taylor instability in an ablation front.

where $A_T = (1 - r_D)/(1 + r_D)$ is the Atwood number, $v_2$ is the ablation velocity, and $\phi_0$ is the fraction of the energy flux driving the ablation that is transported up to the front by thermal conduction.$^{12}$ It is in general $\phi_0 \leq 1$ but in the case of ablation directly driven by ion beams it could be $\phi_0 \ll 1$. Actually, numerical simulations seems to indicate that for the latter case it is $\phi_0 \approx 0.3$. $^{26}$ The density jump $r_D$ is given by the particular mechanism of energy transport, and by taking $r_D \ll 1$, we can write:

$$r_D \approx (nkL_2)^n,$$

where $L_2$ is the characteristic length of the density/temperature gradient in the corona region ($y > 0$) close to the ablation front. For ablation driven by electronic thermal conduction it turns out $n = 2/5$, $^{16-21}$ and for ion beam driven ablation it is $n = 1/3$. $^{12,13,27-29}$ The corresponding expressions for $L_2$ in each case can be found in Refs[12,13,27,28] and are not necessary here. Since there is little difference between the expressions for $r_D$ for these two rather opposite (diffusive and collisional) mechanisms of energy deposition, we will assume that a similar exponent $n$ will result in a situation in which both mechanisms are present. For simplicity we will take $n = 1/3$ without losing generality in the conclusions.$^{13}$

On the other hand, in order to study the dynamic stabilization of ablative RTI, we take:

$$G(t) = g + b\Gamma(\omega t) ; \quad b = \omega^2 A ,$$

where $g$ is the background acceleration, and $\Gamma(\omega t)$ is a periodic function that oscillates with frequency $\omega$ and amplitude $A$. Here, we will consider that the oscillatory acceleration
consists in a general square wave of the form (Fig.1):

\[ b\Gamma(\omega t) = \begin{cases} 
+b_c & \text{if } 2m\pi \leq \omega t \leq 2m\pi + c \\
-b_d & \text{if } 2m\pi + c \leq \omega t \leq 2(m+1)\pi 
\end{cases} , \quad (4) \]

where

\[ c + d = 2\pi , \quad b_c = b_d , \quad (5) \]

The latter conditions are imposed to ensure that the average acceleration of the front is \( < G(t) > = g \). By changing \( c \) (or \( d \)) we can study the effect of the waveform on the performance of the dynamic stabilization.\(^{15}\)

III. THE DISPERSION RELATION

For determining the boundaries of the stability regions and the instability growth rate we must solve Eq.(1) together with Eqs.(2) to (5). Introducing convenient dimensionless variables:

\[ \tau = \omega t ; \quad x = \xi / \xi_0 \quad , \quad (6) \]

where \( \xi_0 = \xi(t = 0) \) is the initial perturbation amplitude, we get the dimensionless version of Eqs.(1) and (4):

\[ \ddot{x} + 2D\dot{x} + [K^2 - \beta\Gamma(\tau)]x = 0 \quad , \quad (7) \]

\[ \beta\Gamma(\tau) = \begin{cases} 
+\beta_c & \text{if } 2m\pi \leq \tau \leq 2m\pi + c \\
-\beta_d & \text{if } 2m\pi + c \leq \tau \leq 2(m+1)\pi 
\end{cases} , \quad (8) \]

where we have used the following definitions:

\[ D = \frac{2kv_2}{\omega} ; \quad K^2 = \frac{\phi_0k^2v_2^2}{\omega^2 r_D} - \frac{k g}{\omega^2} ; \quad (9) \]

\[ \beta_c = \frac{k b_c}{\omega^2} ; \quad \beta_d = \frac{k b_d}{\omega^2} . \quad (10) \]

Since \( \Gamma(\tau) \) with is a periodic function, Eq.(7) results to be a typical damped Hill’s equation that can be treated by means of the Floquet theory.\(^{30}\) In addition, in the previous equations we have already considered that \( r_D << 1 \) and \( A_T \approx 1 \). As in Ref.(13), we also introduce the dimensionless wave number \( \kappa \) and frequency \( \varpi \) as follows:

\[ \kappa = \frac{k v_2}{g} ; \quad \varpi = \frac{\omega v_2}{g} , \quad (11) \]
Thus, the density jump given by Eq.(2) with \( n = 1/3 \) reads:

\[
    r_D \approx \frac{\kappa^{1/3}}{(3Fr_2)^{1/3}}; \quad Fr_2 = \frac{v^2}{gL_2},
\]

where \( Fr_2 \) is the Froude number and it is the characteristic parameter of the steady state ablation front. In the same manner Eqs.(9) and (10) are written in terms of \( \kappa \) and \( \varpi \) as follows:

\[
    D = \frac{2\kappa}{\varpi^2}; \quad K^2 = \frac{\kappa}{\varpi^2} \left[ \left( \frac{\kappa}{\kappa_c} \right)^{2/3} - 1 \right]; \quad \kappa_c = \frac{r_D(\kappa_c)}{\phi_0} \approx \frac{1}{\phi_0^{3/2}(3Fr_2)^{1/2}}
\]

\[
    \beta_c = \frac{\kappa}{\varpi^2} b_c; \quad \beta_d = \frac{\kappa}{\varpi^2} b_d; \quad \frac{\beta_d}{\beta_c} = \frac{c}{d}.
\]

As it was already noticed in Ref.[13], the dimensionless cut-off wave number \( \kappa_c \) given by Eq.(13) is the parameter required to characterize the RTI in the non-oscillating ablation front \((b_c = b_d = 0)\)

With the transformation \( y = x \exp(D\tau) \) usual in dealing with Hill’s equations, Eq.(7) reads:

\[
    \ddot{y} + \left[ K^2 - D^2 - \beta \Gamma(\tau) \right] y = 0.
\]

According to the Floquet theory the solutions of Eq.(15) have the following general property of translational symmetry:

\[
    x(\tau + 2\pi) = \chi x(\tau) \quad \text{or} \quad y(\tau + 2\pi) = \chi e^{2\pi D} y(\tau),
\]

where \( \chi = e^{2\pi \eta} = e^{\pm i m \pi + 2\pi (\sigma + D)} \) or \( i 2\pi \eta = \pm i m \pi + 2\pi (\sigma + D) \).

By multiplying both sides of Eq.(16) by \( \exp[-(i\eta - D)(\tau + 2\pi)] \) and taking into account Eq.(17) it follows that the solutions of Eq.(7) have the following general form:

\[
    x(\tau) = \pm P(\tau) e^{\sigma \tau}; \quad \sigma = \frac{\gamma}{\omega},
\]

where \( P(\tau) \) is a periodic function and \( \gamma \) is the instability growth rate.

For our particular case of the general two-step periodic oscillation \( \Gamma(\tau) \) given by Eq.(8), Eq.(15) can be easily solved in the regions where \( \Gamma(\tau) \) is constant. Then, by imposing the corresponding matching conditions for the solutions \( y(\tau) \) and its derivative \( \dot{y}(\tau) \) at \( \tau = 2m\pi \)
and \( \tau = 2m\pi + c \), and taking into account the translational symmetry of the solutions given by Eq.(16), we can get the dispersion relation\(^{15,30}\)
\[
\cosh(\lambda_d d) \cosh(\lambda_c c) + \frac{\lambda_c^2 + \lambda_c^2}{2\lambda_d \lambda_c} \sinh(\lambda_d d) \sinh(\lambda_c c) = \mp \cosh[2\pi(\sigma + D)] ,
\]
(19)
where:
\[
\lambda_c^2 = D^2 - K^2 + \beta_c ; \quad \lambda_d^2 = D^2 - K^2 - \beta_d ,
\]
and we have taken into account that \( \cos(2\pi \eta) = \pm \cosh[2\pi(\sigma + D)] \). As it could be expected, this dispersion relation is the same as the resulting for the Schrödinger equation when the well known Kronig-Penney periodic potential, consisting in rectangular sections, is considered.\(^{31}\)

Eq.(19) is an explicit expression for the dimensionless growth rate \( \sigma \) as a function of \( \kappa \) for given values of the front parameter \( \kappa_c \) and of the modulation parameters \( b_d/g \) (or \( b_c/g \)) and \( \varpi \), for any type of square wave characterized by the parameters \( c \) and \( b_d \), as presented in Eq.(5). It is not difficult to see that in absence of modulation it is \( b_c = b_d = 0 \) and from Eq.(19) one recover the growth rate for ablative RTI:
\[
\sigma = \sqrt{D^2 - K^2} - D ,
\]
(21)

A. Symmetric square wave \((c = d = \pi)\)

With the purpose to evaluate the effect of different modulations on the dynamic stabilization of the front acceleration we start, for comparison, with the analysis of the case of a perfectly symmetric square wave: \( c = d = \pi \) and \( b_c = b_d \equiv b \). Then, by putting \( \sigma = 0 \) in Eq.(19) we can find the expressions for the positive and negative branches determining the upper and lower limits of stability in the form of an implicit function \( b/g \) of the dimensionless wave number \( \kappa \). Differently to the sequence of Dirac deltas studied in Ref.[13], here both branches result to be multivalued functions of \( \kappa \) with infinite solutions. In this case, the lower limit of stability is given by the lowest of the solutions for the negative branch. All the other solutions corresponding both to the positive and negative branches consist of infinite closed regions of instability limited by lobed curves as it is shown in Fig.2 for \( c = d = \pi \). These lobes determine the upper limit of the stability region. Due to the implicit character of Eq.(19), the solutions must be found numerically by mean of a simple iteration process. Nevertheless, some limits useful for the discussion of the results can be obtained analytically.
FIG. 2. Dimensionless driving accelerations $b/g$ for marginal stability as functions of the dimensionless wave number $\kappa$ for a symmetric square wave driving (SSW) and for: a) $\kappa_c = 0.3$, $\varpi = 1$; b) $\kappa_c = 0.3$, $\varpi = 0.05$; c) $\kappa_c = 5$, $\varpi = 1.5$; d) $\kappa_c = \infty$, $\varpi = 1$. Red and blue curves correspond to the positive and negative branches of Eq.(19), respectively.

For relatively large values of $\varpi$ the upper limit of stability is given by the first lobe from the bottom (in red color in Fig.2). For this lowest lobe, we get the following asymptotic behavior of the modulations amplitudes $b_d/g)_+$ and $b_c/g)_+$ for $\kappa \ll 1$:

$$\frac{b_d}{g}_+ = \frac{b_c}{g}_+ \frac{c}{d} \approx 2 \sqrt{2 \frac{\varpi^2}{cd^3}} \kappa \left[ \sqrt{1 + \frac{(d - c)^2}{2c^2d^3}} - \frac{d - c}{\sqrt{2c^2d^3}} \right]$$  \quad (\kappa \ll 1) . \quad (22)

For the rest of the higher lobes (successively in blue and red in Fig.2), we find that they also go as $1/\kappa$ for $\kappa \ll 1$:

$$\frac{b_d}{g}_\pm = \frac{b_c}{g}_\pm \frac{c}{d} \approx f_\pm(d/c, b_d/g) \frac{\varpi^2}{\kappa d} \quad (\kappa \ll 1) , \quad (23)$$

where $f_\pm(d/c, b_d/g)$ is a multivalued function of $b_d/g$ that for $\kappa << 1$ takes only discrete values. As the dimensionless frequency $\varpi$ decreases, the number of lobes of the upper limit increases and they compact on an envelope curve which can be obtained from Eq.(19) by
taking the limit $\varpi << 1$ [see curve b) in Fig.2]:

$$\left( \frac{\nu_d}{g} \right)_\pm = \frac{\nu_c}{g} \frac{c}{d} \approx \left[ 4 \left( \frac{4\pi^2}{c^2} - 1 \right) \kappa + \left( \frac{\kappa}{\kappa_c} \right)^{2/3} - 1 \right] \frac{c}{d} \ (\varpi << 1) \ . (24)$$

However, this limit presents the unphysical feature that perturbation wave numbers smaller than $\kappa_c$ can still be stabilized when $\varpi << 1$ and this is a consequence of the assumptions of linear stability and incompressibility that underlies to Eq.(1). In fact, Eq.(19) considers as mathematically stable all the periodic solutions with a bounded average value on a period even when, physically, the maximum amplitude could be large enough to make the instability enters in the non-linear regime. Such an information is, of course, not present in the model and, therefore, the limit of $\varpi << 1$ yields rather unphysical results. In fact, we will see later that this limit is never achieved when compressibility effects are included, so that it has no practical interest.

In the same manner we can get the asymptotic behavior of the lower limit $b_d/g_-$ for small and large values of $\kappa$. For $\kappa << 1$ we get:

$$\left( \frac{\nu_d}{g} \right)_- = \frac{\nu_c}{g} \frac{c}{d} \approx 2\pi \sqrt{\frac{2}{\kappa c^3}} \frac{\varpi}{\sqrt{\kappa}} \ (\kappa << 1) \ . (25)$$

Instead, for large values of $\kappa$ ($\kappa >> 1$ or $\kappa \to \kappa_c$) Eq.(19) yields the following expression:

$$\left( \frac{\nu_d}{g} \right)_- = \frac{\nu_c}{g} \frac{c}{d} \approx 4 \left( \frac{\kappa [1 - (\kappa / \kappa_c)^{2/3}]}{1 + (\varpi / 2\pi \kappa)[(d/c - 1)^2 - 1]} \right) \frac{c}{d} \ . (26)$$

As one can see from the previous equation, the lower limit has in general a maximum $\kappa_{max}$ for the largest values of $\kappa$ [Fig.2 b) and c)]:

$$\kappa_{max} \approx \left( \frac{3}{5} \right)^{3/2} \kappa_c \left[ 1 - \left( \frac{5}{3} \right)^{1/3} \frac{\varpi}{2\pi \kappa_c} \right]^3 \ . (27)$$

We can extract an important conclusion from this result. In fact, for $\kappa_c \to \infty$, that is, in absence of transport by thermal conduction, dynamic stabilization of the ablation front results to be impossible since only a restricted range of wave numbers can be stabilized [Fig.2 d)]. This behavior is the analogous to the one found for Newtonian fluids in absence of surface tension,$^{10,11}$ but it contrasts with our previous conclusion obtained in Ref.[13] for a driving consisting in a sequence of Dirac deltas. Such an earlier result seems to be just an artifact of the Dirac deltas modulation and, in general, some minimum fraction of the energy flux driving the ablation process must be transported by thermal conduction for
making dynamic stabilization possible. This point is of particular relevance for ICF directly driven by ion beams in the framework of the scenario recently considered by Logan et al.\textsuperscript{14} In fact, for ablation directly driven by ion beams, most of the energy is transported by classical Coulombian collisions, and thermal conduction may be absent or just be a small fraction of the total energy flux. Nevertheless, as we have already mentioned, numerical simulations show that around one third of the energy flux could be transported by thermal conduction which may be sufficient for making possible dynamic stabilization of an ablation front directly driven by ion beams.

FIG. 3. Stability regions corresponding to a symmetric square wave driving (SSW) for $\kappa_m = 0.10$, 0.15, and 0.20, and for two values of the cut-off wave numbers: a) $\kappa_c = 0.3$; b) $\kappa_c = 5$.

As in Ref.[13], we obtain the minimum value of $b/g$ that is required for stabilizing all the wave numbers $\kappa \geq \kappa_m$, with $\kappa_m < \kappa_c$, by taking the corresponding value of the lower limit for $\kappa = \kappa_m$, $b(\kappa_m)/g$, from curves as those shown in Fig.2 for different values of $\varpi$. But if $b(\kappa_m) < b(\kappa_{max})$, with $\kappa_{max}$ given by Eq.(27), then we take the latter as the minimum required value of $b/g$. Such a minimum is referred as the lower boundary of stability. In the same manner, the upper limit in Fig.2 has a minimum, for a given value of $\varpi$, that determines the maximum value of $b/g$ required for the dynamic stabilization, and it is referred hereafter as the upper boundary of stability. In order to have a finite region of stability we need that the lower boundary be less than the upper boundary, and thus, we can construct the stability charts in the form of a function $b/g$ of the dimensionless frequency $\varpi$, for a given value of the parameter $\kappa_c$ and chosen values of the minimum wave number $\kappa_m$ that we want to stabilize. In Fig.3 we present two typical stability diagrams for $\kappa_c = 0.3$ and $\kappa_c = 5$, for
the same values of $\kappa_m$: 0.10, 0.15 and 0.20. For the case with $\kappa_c = 0.3$, the lower boundary is determined essentially for the the value $b(\kappa_m)$, and we can see that for the largest values of $\kappa_m$ (but $\kappa_m < \kappa_c$), we have the anomalous finite stability region for $\varpi << 1$ that we have previously discussed. For the case $\kappa_c = 5$ such a region is not present since the lower boundary in now fixed by $b(\kappa_{max})$. The general qualitative features are the same as for the sequence of Dirac deltas of Ref.[13]. That is, the smaller the value $\kappa_m$ of the minimum wave numbers that we want to stabilize for a given value of $\kappa_c$, the larger is the relative amplitude $b/g$ and the larger is the dimensionless frequency $\varpi$ required for stabilization. However, important quantitative differences are observed that are better highlighted when the compressibility effects are considered.

Compressibility effects play a fundamental role in the generation of relatively high modulation amplitudes $b/g$ with the modest modulation in the driving pressure $\Delta p/p$ that is possible in an ablation front since ablation pressure can push the front but cannot pull it. In fact, dynamic stabilization of an ablation front can work because the layers of the unablated material close to the ablation surface are compressed and decompressed by the ablation pressure generating a local acceleration $b$ that can be considerably larger than the background acceleration $g$.\cite{5,6,13} Following the same phenomenological approach as in Ref.[13], we have:

\begin{equation}
\kappa_m^{-1} \approx \Delta y \approx c_s^2 \omega^{-1} ; \quad \frac{b}{g} = \frac{\Delta p}{p} \frac{d}{\Delta y} ,
\end{equation}

FIG. 4. Stability regions corresponding to a symmetric square wave driving (SSW) for a) $\kappa_c = 0.3$; b) $\kappa_c = 5$. The boundaries are given for $\kappa_m = M_2 \varpi$, and different values of the Mach number $M_2 = 0.04$, 0.06, and 0.10.
where \( c_{s2} \) is the sound speed in the unablated material, \( p = \rho_2 gd \) is the ablation pressure and \( d \) is the shell thickness. Preliminary one dimensional simulations confirm that when a pressure modulation \( \Delta p/p \) is imposed on the front, an average acceleration \( b \) is developed on a region of thickness \( \Delta y \) beneath the ablation surface that is well approximated by Eqs.\( (28) \). In dimensionless units we have:

\[
\kappa_m \approx M_2 \varpi ; \quad \frac{b}{g} = \frac{\Delta p}{p} \frac{\varpi}{M_2},
\]

(29)

where \( M_2 = v_2/c_{s2} \). Previous equations also show that the minimum value of the wave number \( \kappa_m \) is actually dependent on the oscillation frequency \( \varpi \). Then, by introducing it into Eq.\( (19) \) we can see how compressibility affects the stability charts (Fig.\( 4 \)). We noticed first that while the first condition modifies the lower boundary of stability, the second condition affects the upper boundary only for values of \( M_2 \) larger than the ones considered in Fig.\( 4 \) and it may reduce the stability region if \( M_2 \) turns out to be too large. Fig.\( 4 \) shows

\[\text{FIG. 5. Instability growth rates produced by using different types of drivings: a) negative square wave+positive Dirac deltas (SW+D); b) asymmetric square wave (ASW with } c = \pi/4); c) symmetric square wave (SSW); d) no driving (} b = 0); e) symmetric Dirac deltas (SD).}\]
that compressibility determines the minimum frequency \( \varpi \) and the minimum modulation amplitude \( b/g \) that are required for dynamic stabilization for a given value of \( M_2 \).

In addition, compressibility determines the minimum value of the dimensionless wave number \( \kappa_m \) above which the perturbation wave numbers are stabilized, and it results to be weakly dependent of \( M_2 \). Such a value of \( \kappa_m \) represents the best results that can be achieved with a particular type of waveform of the acceleration modulation and it can be used in order to compare the performance of different types of modulations. For instance, in Ref.[13] we have obtained a value of \( \kappa_m \approx 0.08 \) for a sequence of symmetric Dirac deltas (SD) and \( \kappa_c = 0.3 \), which must be compared with the present value of \( \kappa_m \approx 0.06 \) for the symmetric square wave (SSW). A comparison of the respective growth rates can be seen in Fig.5, curves e) and c), respectively, together with the reference case, \( b = 0 \), with no modulation [curve d)]. In comparing the amplitude \( b \) of the SSW, with \( b' \) for the Dirac deltas, we have to take into account that the latter actually represents the average value of the acceleration on a half period \( \pi \) since, of course, the Dirac delta amplitude is infinite.

B. Effects of the asymmetry

Although the considerable 50% reduction of the dynamic cut-off wave number \( \kappa_m \) between the SD and the SSW cases, it is not really a great relative improvement in comparison with the already achieved from the reference case with a cut-off wave number \( \kappa_c = 0.3 \). This indicates that the best performance of dynamic stabilization is not strongly dependent of the particular modulation waveform provided that it is perfectly symmetric. In particular, we should expect similar results for the sinusoidal driving traditionally considered in literature. However, the general square wave described in the previous paragraphs allows for studying the effect of asymmetric modulations just by taking in Eq.(19) \( c \neq d \neq \pi \ (c + d = 2\pi) \) and \( b_d \neq b_c \) but keeping \( b_d d = b_c c \), as indicated by Eq.(5) in order to assure that \( <G(t)> = g \).

It is not difficult to verify that asymmetries for which \( c > d \) perform worst than the opposite case \( c < d \). An extreme example of this was already considered in Ref.[13] for a sequence of negative Dirac deltas and positive square waves \( (c = 2\pi \) and \( d = 0) \) resulting in a reduction of the cut-off wave number \( \kappa_m/\kappa_c = 0.67 \) (for \( \kappa_c = 0.3 \)) that should be compared with the values 0.25 and 0.2 shown in Fig.5 for the SD and the SSW, respectively. Therefore, we will consider here only situations with \( c < d \) and we will take the particular case \( c = \pi/4 \).
FIG. 6. Dimensionless driving accelerations \( b_d/g \) for marginal stability as functions of the dimensionless wave number \( \kappa \) for an asymmetric square wave driving (ASW) and for: a) \( \kappa_c = 0.3, \varpi = 1.5 \); b) \( \kappa_c = 0.3, \varpi = 0.1 \); c) \( \kappa_c = 5, \varpi = 1.5 \); d) \( \kappa_c = \infty, \varpi = 1 \). Red and blue curves correspond to the positive and negative branches of Eq.(19), respectively.

\( (d = 7\pi/4) \) as representative of the general situation. Fig.6 shows the upper and lower limits of stability for some typical cases analogous to the cases presented in Fig.2 for the SSW, and one can see that all the qualitative features are the same, including the facts that a finite value of \( \kappa_c \) is necessary for dynamic stabilization [curve d)] and that stable solutions can be obtained for \( \varpi << 1 \) [curve b)].

From graphs like those of Fig.6 we can obtain as before, the stability charts for given values of \( \kappa_m \) and \( \kappa_c \) (Fig.7), having the same qualitative characteristics as the SSW shown if Fig.3. But in order to appreciate the performance of this asymmetric square wave (ASW) we have to include the compressibility effects. This is done in Fig.8 for two typical cases \( \kappa_c = 0.3 \) and \( \kappa_c = 5 \) and for several values of \( M_2 \). Again we find that the lowest dynamic cut-off wave number \( \kappa_m \) is not strongly dependent on the values of \( M_2 \) and \( \kappa_c \). That is, although some minimum fraction \( \phi_0 \) of the energy flux is required to be transported by thermal conduction in order to assure the possibility of dynamic stabilization, the ASW
FIG. 7. Stability regions corresponding to an asymmetric square wave driving (ASW) for $\kappa_m = 0.10$, 0.15, and 0.20, and for two values of the cut-off wave numbers: a) $\kappa_c = 0.3$; b) $\kappa_c = 5$.

FIG. 8. Stability regions corresponding to an asymmetric square wave driving (ASW) for a) $\kappa_c = 0.3$; b) $\kappa_c = 5$. The boundaries are given for $\kappa_m = M_2 \bar{\omega}$, and different values of the Mach number $M_2 = 0.04$, 0.06, and 0.10.

perform equally well even if such a fraction is relatively small and it produces a dynamic cut-off wavenumber $\kappa_m \approx 0.03$. Of course, smaller values of $\kappa_m$ are possible, provided that $\kappa_c$ is smaller but, as we have already mentioned, smaller is $\kappa_c$ harder is to get a further reduction of the fraction $\kappa_m/\kappa_c$. In Fig. 5 we can see that the resulting growth rate [curve b)] has a maximum that is a factor of two lower than the one corresponding to the reference case ($b = 0$)

Since the performance of dynamic stabilization improves ($\kappa_m$ is reduced) as smaller is the duration of the positive half periods $c$, we can get insight on the best performance that can be expected from dynamic stabilization of an ablation front by considering the extreme
case \( c = 0 \) \((b = 2\pi)\) which results in a sequence of positive Dirac deltas and negative square waves.

1. **Negative square wave + positive Dirac deltas (SW+D)**

The dispersion relation for this case can be obtained from Eq.(19) by taking the limit \( c \to 0 \) and keeping \( b_c c = constant = 2\pi b_d: \)

\[
\cosh(2\pi \lambda_d) + \frac{\pi \beta_d}{\lambda_d} \sinh(2\pi \lambda_d) = \mp \cosh[2\pi (\sigma + D)] , \tag{30}
\]

As in the previous cases we can get the boundaries of marginal stability by putting \( \sigma = 0 \) in the above equation. Some typical cases are represented in Fig.9 that can be compared with Figs. 2 and 6 for the SSW and the ASW, respectively. Although the general features are similar to the previous cases, there is a noticeable difference in the asymptotic behavior of the lower limit of stability for large values of \( \kappa \). From Eq.(30) we get for \( \kappa \gg 1 \) or \( \kappa \to \kappa_c: \)

\[
\frac{b_d}{g} \approx \frac{2\varpi}{\pi} \left\{ e^{\frac{\pi}{2\varpi} \left[ b_d/g - 1 + (\kappa/\kappa_c)^{2/3} \right]} - 1 \right\} . \tag{31}
\]

Then, for any finite value of \( \kappa_c, b_d/g \to 0 \) for \( \kappa = \kappa_c \) without having any maximum as the ones observed for the general square waves. In addition, for \( \kappa_c \to \infty, b_d/g \) goes to the finite value \( b_d/g)_{inf} \) given by Eq.(31) for \( \kappa_c = \infty \). That is, for this case involving Dirac deltas in the modulation acceleration, dynamic stabilization is possible even in absence of thermal conduction as we have already found in Ref.[13] for others Dirac deltas drivings. Therefore, it appears that in general, dynamic stabilization requires a surface tension-like effect producing a cut-off wave number in a similar manner as it happens with Newtonian fluids and that the result found in this regards in Ref.[13] is a mere artifact of the Dirac deltas driving. Then, in realistic situations some fraction of the energy flux that drives the ablation must be transported by thermal conduction in order to make possible the dynamic stabilization of an ablation front.

We can also get the asymptotic behavior of the lower limit \( b_d/g)_- \) for \( \kappa << 1: \)

\[
\frac{b_d}{g} \approx \frac{\sqrt{3}}{\pi} \frac{\varpi}{\sqrt{\kappa}} \quad (\kappa << 1) . \tag{32}
\]

As one can see, in this limit it is \( b_d/g)_- \propto \varpi/\sqrt{\kappa}, \) such as it results for any other type of modulation, either for a general square waves [Eq.(25)] or Dirac deltas,\(^{11,13}\) and also for the
FIG. 9. Dimensionless driving accelerations $b_d/g$ for marginal stability as functions of the dimensionless wave number $\kappa$ for a driving consisting of a square wave + Dirac deltas (SW+D) and for:

a) $\kappa_c = 0.3$, $\varpi = 1.5$; b) $\kappa_c = 0.3$, $\varpi = 0.1$; c) $\kappa_c = \infty$, $\varpi = 0.1$. Red and blue curves correspond to the positive and negative branches of Eq.(30), respectively.

sinusoidal driving.\textsuperscript{7} The particular type of modulation only affects the numerical factor of proportionality.

From the above limits of stability we get again the stability charts. In Fig.10 we present the results including already the compressibility effects that show the best performance that can be obtained with square wave modulations. This case may be indicative of the best results that can be expected with any kind of driving. That is, for $\kappa_c \geq 0.3$ it results $\kappa_m \approx 0.015$ what means a reduction of a factor larger than 20 ($\kappa_m/\kappa_c = 0.05$) in the cut-off wave number. For the same case the maximum growth rate results to be reduced by more than a factor of 3 with respect to the reference case [Fig.5, curve a)]. Of course, as we have already discussed, such large reduction factors in the dynamic cut-off wave number are not held for progressively lower values of $\kappa_c$ since the resulting value of the dynamic wave number $\kappa_m$ seems to be not strongly dependent on $\kappa_c$. For instance, when $\kappa_c$ is as low as $\kappa_c = 0.05$, we still get $\kappa_m \approx 0.015$ and the maximum reduction factor that can be obtained (for the SW+D modulation) is around 3 ($\kappa_m/\kappa_c = 0.3$).
FIG. 10. Stability regions corresponding to a driving consisting of a square wave + Dirac deltas (SW+D) for a) $\kappa_c = 0.3$; b) $\kappa_c = \infty$. The boundaries are given for $\kappa_m = M_2 \bar{\omega}$, and different values of the Mach number $M_2 = 0.04$, 0.06, and 0.10.

IV. CONCLUDING REMARKS

We have studied the problem of the dynamic stabilization of RTI in an ablation front by considering a general square wave for the modulation of the driving acceleration. This approach allows for finding an explicit expression for the instability growth rate and an implicit analytical equation for the dispersion relation. In this manner we can analyze the effect of different modulations just by considering asymmetries in the duration and amplitudes of the modulation half-period.

We have found that asymmetries consisting in a short duration and large positive acceleration followed by long duration and small negative acceleration, perform better than the opposite case. The best performance corresponds to the limiting case when the positive acceleration is a Dirac delta. Such a kind of drivings resemble the picked fence pulses considered in the literature also for stabilization of RTI in the ablation front but based on a different principle. In fact, picked fence pulses have been proposed for controlling RTI in the ablation front by generating an entropy shaping that drives the layers beneath the front on a higher adiabat. In this way, an increase in the ablation velocity is produced that improves the front stability. Such effects are, of course, not included in our analysis of the dynamic stabilization but they could certainly be present in a realistic situation providing a further mechanism of stabilization.
In addition, we have shown that, in general, some minimum fraction of the energy flux that drives the ablation process must be transported by thermal conduction in order to make possible the dynamic stabilization of the ablative RTI. That is, dynamic stabilization in an ablation front behaves in a similar manner to Newtonian fluids for which some minimum surface tension, besides of the viscosity, is required. These analogies suggest the possibility to use Newtonian fluids for surrogate experiments that may help to understand the physics of dynamic stabilization in a much simpler experimental framework.

The results involving Dirac deltas showing that stabilization is possible even when $\kappa_c = \infty$, seems to come out just as an mathematical artifact of the Dirac deltas. However, for applications to ablation fronts directly driven by ion beams it is possible that the small fraction $\phi_0 = 0.3$ observed in the numerical simulations be sufficient for allowing to use dynamic stabilization for controlling RTI.

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