Dynamic stabilization of Rayleigh–Taylor instability in an ablation front

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Dynamic stabilization of Rayleigh–Taylor instability in an ablation front is studied by considering a modulation in the acceleration that consists of sequences of Dirac deltas. This allows obtaining explicit analytical expressions for the instability growth rate as well as for the boundaries of the stability region. As a general rule, it is found that it is possible to stabilize all wave numbers above a certain minimum value $k_m$, but the requirements in the modulation amplitude and frequency become more exigent with smaller $k_m$. The essential role of compressibility is phenomenologically addressed in order to find the constraint it imposes on the stability region. The results for some different wave forms of the acceleration modulation are also presented. © 2011 American Institute of Physics. [doi:10.1063/1.3535400]

I. INTRODUCTION

Rayleigh–Taylor instability (RTI) is of central importance for inertial confinement fusion (ICF) since it is one of the main factors that determine the minimum energy required for achieving ignition and high energy gains. 1 In general, the ablation process used to drive the implosion of an ICF capsule containing the fusion fuel provides a natural method for stabilizing the shortest wavelengths $\lambda$. However, the corresponding minimum stable wave number, or cut-off wave number $k_c=2\pi/\lambda_c$, may not be sufficiently small to ensure the success of the implosion and, in any case, smaller $k_c$ allows thinner shell targets and less input energy required to accelerate it to ignition velocities. These considerations are particularly relevant in the case of inertial fusion directly driven by ion beams that has been recently reproposed by Logan et al. 2 In fact, as it was shown by Piriz et al. 3 and confirmed by LASEX simulations, 4 in such a case the cut-off wave number results to be considerably large as a consequence of the relatively small fraction $\phi_0$ of the beam energy that is transported by thermal diffusion.

Therefore, any method having the potential to mitigate the instability growth rate during the process of implosion is certainly of great importance for ICF. In this framework, dynamic stabilization driven by the vertical vibration of the ablation front has been suggested as a possible method for the stabilization of an ablation front directly driven by ion beams. 5 Dynamic stabilization in the ICF scenario was first proposed in 1977 by Boris, 6 who was able to demonstrate it in two dimensional numerical simulations but could not reach a complete comprehension of the basic mechanisms since at that time the essential physics of the linear phase of the ablative RTI that should serve as the reference case was not yet well understood. In fact, Boris had to use a phenomenological equation for the linear evolution of the perturbation amplitude in which the damping effects of the ablation process were not present. As we will see here, such effects are crucial to the dynamic stabilization of the largest wave numbers $k$.

A second analysis, and probably the most relevant study performed so far, was presented much later in 1993 by Betti et al. 7 As in the previous case, the theoretical treatment of the linear stage of growth of RTI in an ablation front 8–12 was not yet available at that time and for this, Betti et al. based their analysis in an evolution equation of the type of the Takabe fitting formula. 13 Such an approach, in opposition to the Boris study, overestimated the magnitude of the damping effects, producing rather optimistic results.

On the other hand, the previous works have considered a sinusoidal vibration of the ablation front. As noticed by Boris, such a sinusoidal modulation may not necessarily be the optimum one but it was used for considering it analytically the most tractable one. Nevertheless, a sinusoidal modulation leads to a Mathieu equation that, in general, must be solved numerically, and finding the stability regions is a very difficult task. In addition, it prevents obtaining scaling law relationships that are necessary for extracting general conclusions and for the design and interpretation of experiments. In particular, it is of great importance to know what is the minimum perturbation wave number $k_m$ that can be stabilized by a particular type of modulation with the given frequency $\omega$ and amplitude $b$ in the vibration of the acceleration around the background gravity $g$.

Dynamic stabilization of RTI in Newtonian fluids has already been experimentally demonstrated by applying a vertical sinusoidal modulation, 14–16 and these experiments were theoretically analyzed by Troyon and Gruber, 17 who showed the importance of the effects of viscosity and surface tension in determining the region of parameters space for which dynamic stabilization is possible. The recent study by Kawata et al., 18 although aimed at applications to ion beam fusion, does not actually consider RTI in an ablation front but in classical ideal fluids. For such a case, dynamic stabilization is not possible because only a restricted range of perturbation wave numbers can be stabilized. 14–18 Nevertheless, Kawata et al. found a reduction in the perturbation growth.

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Recently, an analysis of the experiments by Wolf et al.\textsuperscript{14-16} points out the close analogies of RTI in Newtonian fluids and in ablation fronts.\textsuperscript{18} In the latter, the equivalent effects of the viscosity and of the surface tension are naturally produced by the mechanism of ablation itself. It has been shown that the essential physics of the RTI dynamic stabilization can be captured by the simplest possible modulation, which is described by a sequence of Dirac deltas representing an impulsive periodic modulation.\textsuperscript{18,19} Such an approach allows finding the general similarity parameters that are actually independent of the particular type of modulation used for the dynamic stabilization. Of course, the stability region is determined by the functions of such parameters that, in general, will depend on the modulation waveform.

However, despite the previously mentioned analogies between RTI in Newtonian fluids and in ablation fronts, the different dependence of the damping and surface tensionlike effects on the perturbation wave number preclude a direct extrapolation of the results obtained for Newtonian fluids to the case of ablation fronts. Furthermore, as noticed by Boris\textsuperscript{9} and Betti et al.,\textsuperscript{7} the compressibility effects play an essential role in the dynamic stabilization of ablation fronts, and as we will see later in this paper, they impose further restrictions to the parameter space of the stability region. Therefore, the dynamic stabilization of RTI in ablation fronts requires a specific study.

Here, we present such a study for the particular case of an ablation front directly driven by ion beams for which this stabilization method can be of special interest.\textsuperscript{2,3,5} Notwithstanding, application of the methods presented here to ablation fronts driven by thermal diffusion or any other diffusive process of energy transport is straightforward and the present results can be easily generalized to those cases. As in the work of Piriz et al.,\textsuperscript{18} we use a modulation in the driving acceleration that consists of a sequence of Dirac deltas. This approach allows us to find complete close solutions for the problem of the dynamic stabilization of RTI in an ablation front. In particular, we get explicit analytic solutions for the growth rate and for the boundaries of the stability region in terms of the parameters of the modulation and of the steady state ablation front. Besides, we can consider different types of sequences in order to evaluate the effect of the different modulations. Here, we will concentrate on two cases: (a) a symmetric impulsive driving consisting of a positive impulse (adding to the background gravity), followed by a negative impulse in the same period, and (b) a negative impulse in one period. More general modulations schemes involving Dirac deltas and/or square driving waves are also analytically tractable and they could be considered in a further study for understanding the optimization process of the wave form of the acceleration modulation.\textsuperscript{20}

### II. THE ANALYTICAL MODEL AND FUNDAMENTAL EQUATIONS

We consider here the same problem as in Ref. 3 in which a perturbed ablation front of zero thickness is placed at \( y = \xi(x, t) \) and it separates two fluids of densities \( \rho_2 \) and \( \rho_1 < \rho_2 \), ahead and behind the ablation front, respectively. We assume a uniform time varying gravitational field \( G(t) \) pointing in the positive \( y \)-axis direction, which is taken in the opposite direction to the density gradient. As it was noticed by Piriz et al.,\textsuperscript{12} we are assuming a sharp boundary model for the ablation front but, provided that the adequate extra information regarding the density profiles at both sides of the front is conveniently incorporated, the model will yield the correct results. As it is already well known, such information is included in the sharp boundary model by considering the self-consistent density jump \( r_p = \rho_1 / \rho_2 \), with \( \rho_1 \) and \( \rho_2 \) taken as the density values at a distance equal to \( k^{-1} \) from the front, behind, and ahead of it, respectively.\textsuperscript{12} Here, in order to obtain the equation of motion of the interface, we follow the method in Ref. 3. Let us briefly summarize the main steps for deriving the required equation. First, we observe that if the ablation front is in equilibrium, then the normal components of the forces per unit surface \( f(y) \) on each side of the interface, and due to one of the fluids \( (\nu=1, 2) \), are exactly balanced and \( f(y) = 0 \). The force on the interface due to the fluid \( \nu \) on each side of it is \( f_{(y)} = \Pi(y) n_{(y)} \), where \( \Pi(y) = - \rho_p \delta_k + \rho_p \delta_k + \rho_p \delta_k \) is the momentum flux density tensor for an ideal fluid. \( \rho_p \) is the thermodynamic pressure of the fluid \( \nu \), \( \delta_k \) is the Kronecker delta, \( n_{(y)} \) is \( \nu \)-component of the fluid velocity in the medium \( \nu \), and \( n_{(y)} \) is the \( \nu \)-component of the unit vector \( u \) directed outward along the normal to the interface.\textsuperscript{21} In equilibrium, we have \( n_{(y)} = - n_{(y)} \), \( p_1 + p_2 u_{(y)} \), where \( u_1 = u_{(y)} \) and \( u_2 = - u_{(y)} \), and the force balance is identically satisfied.

If the interface is not in equilibrium and it has a small sinusoidal perturbation of amplitude \( \xi(x) \) and wave number \( k \), such that the elements which in the equilibrium state would be at \( y = \xi(x) \) are actually at the position \( y = \xi(x) \), then a force imbalance \( \delta f_{(y)} + \delta f_{(y)} \) will exist on the interface. This net force produces the motion of the fluid in the regions close to the interface, at both sides of it, in order to satisfy the Newton second law,\textsuperscript{3,22-24}

\[
\frac{d}{dt} \left[ (m_1 + m_2) \dot{\xi} \right] = \delta f_{(y)} + \delta f_{(y)},
\]

where the dot indicates time derivative and \( m_{(y)} \) is the mass per unit surface of the fluid \( \nu \) that is involved in the motion due to the instability. Since we are dealing with surface modes that decay exponentially from the interface with the characteristic scale \( k^{-1} \), we can consider that in the linear regime only the fluid within this distance is affected by the motion of the interface.\textsuperscript{3,10-12,22-27} Therefore, we have \( m_{(y)} = \rho_{(y)} / k \).

On the other hand, the perturbation in the force on the interface due to the fluid \( \nu \) is \( \delta f_{(y)} = \delta \Pi_{(y)} n_{(y)} = [\delta p_{(y)} + \delta n_{(y)} p_{(y)} + m_{(y)} \delta v_{(y)}] n_{(y)} \), where \( u_{(y)} = \delta v_{(y)} \), \( \delta n_{(y)} \), and \( \delta v_{(y)} \) are the unperturbed velocity, and the perturbations in pressure, mass ablation rate, and velocity of the fluid \( \nu \), respectively. In writing \( \delta f_{(y)} \), we have taken into account that in the linear regime, \( n_{(y)} \sim k \xi \ll 1 \) and \( n_{(y)} \sim 1 \). As noticed in Ref. 3, we have

\[
\begin{align*}
\end{align*}
\]
\[ \dot{\rho}^{(v)} = \rho_0 G \xi , \quad \dot{\rho}^{(1)} = -\dot{\xi}^{(2)} = \dot{\xi} , \quad \dot{\eta}_1 = -\dot{\eta}_2 = \dot{\eta}. \]

(2)

Therefore, Eq. (1) for the evolution of the interface reads as

\[ \frac{\rho_1 + \rho_2}{k} \dot{\xi} = (\rho_2 - \rho_1) g \xi - 4 m \dot{\xi} - \dot{\eta}(v_1 + v_2). \]

(3)

The last term \( \dot{\eta}(v_1 + v_2) \) is the reaction force due to the perturbation \( \dot{\eta} \) in the mass ablation rate that arises when the interface moves through the temperature gradient of the ablation corona. \(^5,12,28\) This term is determined by the particular mechanism driving the ablation process. Actually, Eq. (3) has a more general character and it applies to any RTI process involving flux of mass and momentum through the interface as it would also be, for instance, in the case of miscible fluids with mass diffusion across the interface. \(^29\) Thus, in order to specify a particular problem, it is necessary to introduce extra information. In the case of the ablation front, this information comes from the fundamental property of the ablation front of being an isotherm. Then, the perturbation in the specific internal energy \( \epsilon = k_B T / (\gamma - 1) Am_p \) (where \( k_B \) is the Boltzmann constant, \( T \) is the temperature, \( \gamma = 5/3 \) is the enthalpy coefficient, \( A \) is the mass number, and \( m_p \) is the proton mass) in the corona region close to the front \( y > 0 \) can be written as follows: \(^{12}\)

\[ \delta \epsilon = -\frac{d \epsilon}{d y}. \]

(4)

Now, in order to apply Eqs. (3) and (4) to a particular kind of ablation front, we must define the specific mechanism that drives the ablation process.

Here, we will consider the case of ablation directly driven by ion beams. \(^2,3,30,31\) Thus, we assume an intense beam of ions with mass \( m_0 \), charge number \( Z_p \), and energy \( E = (m_0 / 2) v^2_p \), which interacts with the corona plasma by classical Coulomb collisions. By neglecting backscattering, we can write the following equations for the energy flux \( Q_b \) transported by the ions:\(^{3,30,32}\)

\[ \frac{1}{Q_b} \frac{d Q_b}{d y} = 1 \frac{d E}{E d y} = 1 \frac{12}{L} - \frac{\alpha E^2}{\rho_0 (v_b / v_{th e})}, \]

(5)

where \( v_{th e} = (2 k_B T / m_e)^{1/2} \) is the electron thermal velocity \( (m_e \) is the electron mass), and in addition, we have

\[ \psi(w) = \frac{1}{\sqrt{\pi}} \left[ \int_0^w e^{-x^2} dx - w e^{-w^2} \right], \]

(6)

\[ \alpha = \frac{A m_p m_e}{Z m_p} \frac{1}{2 \pi Z_p e^4 \ln \Lambda_b}, \]

where \( Z \) is the charge numbers of the ablation plasma and \( \ln \Lambda_b \) is the Coulombian logarithm and it will be taken as a constant \( (\ln \Lambda_b = 8) \). In Eq. (6), \( \psi(w) = 1 \) for \( w \gg 1 \) and \( \psi(w) = (4 / 3 \sqrt{\pi}) w^3 \) for \( w \ll 1 \). In the region close to the ablation front, the Mach number of the corona plasma \( (y > 0) \) is \( M_1 \gg 1 \) and the corona profiles can approximately be described by the following equations: \(^{3,30,32}\)

\[ \dot{\rho} = \rho_0 G \xi , \quad \dot{\rho}^{(1)} = -\dot{\xi}^{(2)} = \dot{\xi} , \quad \dot{\eta}_1 = -\dot{\eta}_2 = \dot{\eta}. \]

(7)

\[ p = p_0 = \text{const}, \]

(8)

\[ \epsilon = \frac{v}{v_2} = \frac{\rho_2}{\rho}, \]

(9)

where \( n_b \) is the particle density of the ion beam and \( \epsilon_2 \) is the specific internal energy of the medium ahead de front \( (y < 0) \). From the previous equations, we can get

\[ w = \frac{v_b}{v_{th e}} = \left[ \frac{m_e}{(\gamma - 1) Am_p n_b \mu_b} \right]^{1/2} \left( \frac{\theta - 1}{\theta} \right)^{1/2} \]

(10)

\[ \theta = \epsilon / \epsilon_2 \]. Therefore, by assuming \( \theta \gg 1 \) in Eq. (10), we can take \( \psi(w) \approx \psi(w_0) \), and from Eqs. (9) and (10), we generalize the expression for the density profile close to the ablation front given by Piriz et al. \(^3\) by obtaining an equation that is approximately valid for any regime of the beam velocity \( v_b \),

\[ \frac{\gamma}{L_2} = \frac{1}{6} + \frac{1}{3} \left( \frac{\rho_2}{\rho} \right)^3 - \frac{1}{2} \left( \frac{\rho_2}{\rho} \right)^2, \]

(11)

\[ L_2 = \frac{\alpha E^2}{\rho_2 \psi(w_0)}, \quad E_2 = \frac{\gamma \dot{m} \epsilon_2}{\mu_b}. \]

(12)

As in Ref. 3, from Eqs. (7)–(9), we get that the perturbation in the mass ablation rate required in Eq. (3) is \( \dot{\eta} / \dot{\rho} \ll k_\xi \ll 1 \), and the last term in Eq. (3) turns out to be negligible in comparison with the other terms. However, by allowing that some part \( Q_T \) of the beam energy flux be transported by thermal conduction so that \( \phi_b = Q_T / (Q_T + Q_b) \), we can write for the perturbation in the mass ablation rate a more general expression, \(^3\)

\[ \frac{\dot{\eta}}{\dot{\rho}} = \phi_b k \xi. \]

(13)

Of course, when some fraction \( \phi_b \) of the energy flux is transported by thermal conduction, the density profile in the corona region that we have obtained in Eq. (11) should be modified consistently. However, for simplicity, we will assume that this profile is still valid for estimating the dependence of the self-consistent density jump \( r_D \) on the perturbation wave number \( k \). We will see later that this assumption does not represent a significant loss of generality. Therefore, introducing Eq. (13) into Eq. (3), we obtain the following equation of motion for the interface: \(^{3,8,12}\)

\[ \ddot{\xi} + \frac{4 k v_2}{1 + r_D} \dot{\xi} + \left[ \phi_b \frac{k^2 v_2^2}{r_D} - A_\xi k G(t) \right] \dot{\xi} = 0, \]

(14)

where \( A_\xi = (1 - r_b) / (1 + r_D) \) is the Atwood number and, as we have already discussed, \( r_D \) can be calculated from Eq. (11) by setting \( \rho_1 = \rho (y = k^2) \).
\[ \frac{1}{k L_2} \approx \frac{1}{3} \left( \frac{1}{r_{D}^{2}} - 1 \right) - \frac{1}{2} \left( \frac{1}{r_{D}^{2}} - 1 \right). \]  

(15)

For simplicity, we will take \( r_D \approx 1 \) and, thus, it turns out \( A_T = 1 \) and

\[ r_D \approx \left( \frac{k L_2}{3} \right)^{1/3}. \]  

(16)

For the case of a corona driven purely by electronic thermal conduction,\(^{8,10,12}\) and within the same limit of \( r_D \approx 1 \), we would get \( r_D = (5k L_2)^{2/5} \), which is not significantly different. Therefore, we can expect that in a general case for which both mechanisms, thermal conduction and collisional beam deposition, are present, Eq. (16) will still represent a good approximation to the density jump \( r_D \).

For our purpose of studying the dynamic stabilization of RTI, we take

\[ G(t) = g + b \Gamma(\omega t), \quad b = \omega^2 A, \]  

(17)

where \( \Gamma(\omega t) \) is a periodic function that oscillates with a frequency \( \omega \) and \( A \) is the amplitude of the oscillatory motion. We will assume that the oscillatory acceleration consists of a sequence of impulses of the form

\[ b \Gamma(\omega t) = b_1 \delta(\omega t - 2m\pi) - b_2 \delta(\omega t - (2m + 1)\pi), \]  

(18)

where \( \delta(\tau) \) are Dirac deltas, \( A_1 = b_1 / \omega^2 \) and \( A_2 = b_2 / \omega^2 \) are the amplitudes of the positive and negative impulses, respectively, and \( m \) is an integer. We will consider two cases: (i) a symmetric driving with \( b_1 = b_2 \) and (ii) an asymmetric driving with \( b_1 = -b_2 \) and \( b_2 = 0 \). With this form for \( \Gamma(\omega t) \), Eq. (14) is a damped Hill’s equation that can be handled by means of the Floquet theory in order to find the instability growth rate and the charts of marginal stability.\(^{33}\)

**III. THE DISPERSION RELATION**

To find the instability growth rate and the boundaries of the stability regions, we must solve Eq. (14) together with Eqs. (16)–(18), and for this, it is convenient to introduce the following dimensionless variables:

\[ \tau = \omega t, \quad x = \xi / \xi_0, \]  

(19)

where \( \xi_0 \) is the initial perturbation amplitude (at \( t=0 \)). Thus, Eqs. (14)–(18) read as

\[ \ddot{x} + 2D \dot{x} + \left[ K^2 - \beta \Phi(\tau) \right] x = 0, \]  

(20)

\[ \beta \Phi(\tau) = \beta_1 \delta(\tau - 2m\pi) - \beta_2 \delta(\tau - (2m + 1)\pi), \]  

(21)

where we have introduced the following definitions:

\[ D = \frac{2k v_2}{\omega}, \quad K^2 = \frac{\phi k v_2^2 }{\omega g r_D} - \frac{k g}{\omega^2}, \]  

(22)

\[ \beta_1 = k A_1, \quad \beta_2 = k A_2. \]  

(23)

In the previous equations, we have already considered that \( r_D \approx 1 \) and \( A_T \approx 1 \). By further introducing the dimensionless wave number \( \kappa \) and frequency \( \sigma \) as follows:

\[ \kappa = \frac{k v_2^2}{g}, \quad \sigma = \frac{\omega v_2}{g}, \]  

(24)

the density jump given by Eq. (16) reads as

\[ r_D = \frac{\kappa^{1/3}}{(3 Fr_2)^{1/3}}, \quad Fr_2 = \frac{v_2^2}{g L_2}, \]  

(25)

where \( Fr_2 \) is the Froude number and it is the characteristic parameter of the steady state ablation front. In the same manner, Eqs. (22) and (23) are written in terms of \( \kappa \) and \( \sigma \) as follows:

\[ D = \frac{2\kappa}{\sigma}, \quad K^2 = \left[ \frac{\kappa}{\kappa_c} \right]^{2/3} - 1, \]  

(26)

\[ \kappa_c = \frac{r_D(\kappa_c)}{\phi_0} = \frac{1}{\phi_0^{3/2} (3 Fr_2)^{1/2}}, \]  

\[ \beta_1 = \frac{\kappa b_1}{\sigma^2}, \quad \beta_2 = \frac{\kappa b_2}{\sigma^2}. \]  

(27)

In Eq. (26), we have introduced the dimensionless cut-off wave number \( \kappa_c \), which is the only parameter required to characterize the RTI in nonoscillating ablation front (\( b=0 \)). As it can be appreciated, \( \kappa_c \) is determined by the Froude number \( Fr_2 \) and by the fraction \( \phi_0 \) of the energy flux that is transported by thermal diffusion, and it becomes larger as the value of \( \phi_0 \) gets smaller.

Performing the usual transformation \( y = x \exp(D \tau) \), we get

\[ \ddot{y} + \left[ K^2 - D^2 - \beta \Phi(\tau) \right] y = 0. \]  

(28)

According to the Floquet theory, the solutions of Eq. (20) have the following general form:

\[ x(\tau) = P(\tau) e^{\gamma \tau}, \quad \sigma = \frac{\gamma}{\omega}, \]  

(29)

where \( P(\tau) \) is a periodic function and \( \gamma \) is the instability growth rate. In addition, these solutions have the general property of translational symmetry,

\[ x(\tau + 2\pi) = \chi x(\tau), \quad y = (\tau + 2\pi) = \chi e^{2\pi D} y(\tau), \]  

(30)

where

\[ \chi e^{2\pi D} = e^{i 2 \pi \eta} = e^{\pm i m \pi + 2\pi(\sigma + D)} \]  

or \( i 2 \pi \eta = \pm i m \pi + 2\pi(\sigma + D) \).

Here, \( \eta \) is the so-called characteristic exponent, and for the cases of marginal stability, it turns out that \( \sigma = 0 \).\(^{20,33}\)

**A. Symmetric driving**

Let us first consider the case of the symmetry modulation that results from Eq. (21) by taking \( \beta_1 = \beta_2 = \beta = (\kappa / \sigma^2)(b/g) \). In such a case, Eq. (28) can be easily solved in the regions where \( \Gamma(\tau) = 0 \) and by imposing the corresponding matching conditions at \( \tau = 2m \pi \) and \( \tau = (2m + 1) \pi \). Thus, we obtain the dispersion relation.\(^{18,33}\)
\[
\beta^2 = 4(K^2 - D^2) \frac{\cos 2\pi \sqrt{K^2 - D^2} - \cos 2\pi \eta}{1 - \cos 2\pi \sqrt{K^2 - D^2}},
\]
where, taking into account Eq. (31), we have
\[
\cos 2\pi \eta = \pm \cos 2\pi (\sigma + D).
\]

Using Eqs. (26) and (27) (with \(b_1 = b_2 = b\)), we can rewrite the previous equation for the dispersion relation in the following manner:
\[
\frac{b}{g} = 4\sigma \left[ \left(1 - \frac{\kappa^2}{D^2} \right) \cosh 2\pi D / \left(1 - \kappa^2 / D^2 \right) \pm \cosh 2\pi (\sigma + D) \right]^{1/2},
\]
where
\[
D = \frac{2\kappa}{\omega}, \quad \frac{K^2}{D^2} = \left(\kappa / \kappa_0\right)^{2/3} - 1 / 4\kappa.
\]

By setting \(\sigma = 0\) in Eq. (34), we obtain explicit analytic expressions for the upper (plus sign) and lower (minus sign) limits of stability in the form of functions \(b/g\), with \(\sigma\) and \(\kappa\) as parameters. As we have already discussed, the latter is the only parameter necessary to characterize the nonoscillating ablation front.

The first thing that can be noticed from Eqs. (34) and (35) is that the damping term \(D\) in the previous equations is essential for making the dynamic stabilization possible and in the case it is absent, like in the problem originally considered by Boris, only a narrow range of wave numbers will be stabilized. In fact, setting \(D = 0\) in Eq. (34), we can obtain the limits of marginal stability (for \(\sigma = 0\)), with \(\kappa_c\) and \(\sigma\) as parameters. In Fig. 1, we show a particular case for \(\kappa_c = 1\) and \(\sigma = 0.5\) that is representative of the general situation when \(D = 0\). In this case, only the wave numbers within the interval \(\kappa_c < \kappa < \kappa_0\) are stabilized, where \(\kappa_0\) is the value for which \(2\pi \kappa (\kappa_0 / \kappa) = \pi\), that is, \(4\kappa_0 / \sigma^2 [\left(\kappa_0 / \kappa\right)^{2/3} - 1] = 1\). This interval of stable wave number becomes even narrower with larger \(b/g\). In this sense, dynamic stabilization of RTI in an ablation front behaves in the same manner as in a Newtonian fluid.

In general, for \(D \neq 0\), the upper (“+”) and the lower (“−”) limits of marginal stability are like the ones shown in Fig. 2 for two typical cases: \(\kappa_c = 0.3\), \(\sigma = 0.8\) and \(\kappa_c = 1\), \(\sigma = 0.5\). The upper limit has a minimum, and beyond it for \(\kappa \gg 1\), it grows exponentially in the following manner:
\[
\frac{b}{g} = 4\sigma [1 + e^{(\pi/2\sigma)[(\kappa / \kappa_0)^{3/2} - 1]}]^{1/2} \quad (\kappa \gg 1),
\]
while for \(\kappa \ll 1\), it goes as
\[
\frac{b}{g} = \frac{2\sigma^2}{\pi \kappa} \quad (\kappa \ll 1).
\]

Therefore, dynamic stabilization is only possible for values of \(b/g\) below the minimum of the upper limit. That is, such a minimum determines the maximum value \(b/g\) that can be used and, hereafter, it will be referred to as the upper boundary of stability.

On the other hand, for \(\kappa \ll 1\), the lower limit decreases monotonically from infinite as follows:

![FIG. 1. Dimensionless driving acceleration b/g for marginal stability as a function of the dimensionless wave number \(\kappa\). Upper and lower limits are shown for \(\kappa_c = 1\) and \(\sigma = 0.5\) and for the case \(D = 0\).](image1)

![FIG. 2. Dimensionless driving accelerations b/g for marginal stability as functions of the dimensionless wave number \(\kappa\) for \(\kappa_c = 0.3\), \(\sigma = 0.8\) and for \(\kappa_c = 1\), \(\sigma = 0.5\). Symbols “+” and “−” denote the stability upper and lower limits, respectively.](image2)
and, when \( \kappa \) approaches the cut-off wave number \( \kappa_c \), it goes to zero in the following manner:

\[
\frac{b}{g} = 4\sigma[1 - e^{(\pi/2\sigma)(\kappa/\kappa_m)^2}]^{1/2} \quad (\kappa \to \kappa_c).
\]

The lower limit determines the minimum value \( b/g \) that is required in order to dynamically stabilize all wave numbers \( \kappa \geq \kappa_m \), with \( \kappa_m < \kappa_c \). Such a minimum value \( b/g \) will be referred hereafter as the lower boundary of stability. Clearly, in order to have a region of stability in the parameter space of the problem, it is necessary that \( b/g \) be less than \( b/g \) for chosen values of \( \kappa_m \). In this manner, we can construct the stability charts of \( b/g \) as a function of \( \sigma \) that give the stability region for a given value of \( \kappa_m \) and for chosen values of the minimum wave number \( \kappa_m \) that we want to stabilize. That is, the stability charts are constructed by taking the minimum value of the positive branch of Eq. (34) and representing it as a function of \( \sigma \) for a given value of \( \kappa_m \) in order to find the upper boundary. For the lower boundary, instead, we set \( \kappa = \kappa_m \) in the negative branch of Eq. (34) (with \( \sigma = 0 \)) and again we represent it as a function of \( \sigma \) for a given value of \( \kappa_m \). Typical charts are shown in Figs. 3–5 for three different values of \( \kappa_c \): 0.3, 1, and \( \infty \). This figures show the general behavior: the smaller the minimum value \( \kappa_m \) of the wave numbers that we want to stabilize for a given value of \( \kappa_c \) the larger the relative amplitude \( b/g \) and the larger frequency \( \sigma \) of the modulation are necessary for stabilization. In the same manner, the smaller \( \kappa_c \), the more difficult to reduce \( \kappa_m \) to a given fraction of \( \kappa_c \), thus requiring higher values of \( b/g \) and \( \sigma \). However, in Fig. 5 we see that for the case \( \kappa_c = \infty \), which corresponds to the case \( \phi_0 = 0 \) in Eq. (14), it is still possible to stabilize the largest wave numbers, although as in the previous cases, it turns out to be more difficult with smaller desired values of \( \kappa_m \). This result differs from the corresponding case for Newtonian fluids, for which it is not possible to stabilize the wave numbers larger than a given value \( \kappa_m \) if surface tension effects are not present. 18

Since the goal of dynamic stabilization is not only to stabilize the wave numbers larger than a given value \( \kappa_m \)
(which should be as small as possible) but also to reduce the maximum growth rate of the unstable modes, it is useful to calculate the growth rate $\gamma (\gamma v_2 / g = \sigma \omega)$ as a function of $\kappa$ for given values $\kappa_c$ and $\sigma$ and for several values of $b / g$. In addition, it could be imagined that by increasing the value of $b / g$ above the upper boundary, in the region known as the one corresponding to parametric instabilities, the instability growth rate could be lower than in the reference case ($b / g = 0$). For this case, the instability growth rate must also be explored in such a region.

The instability growth rate can explicitly be obtained by solving Eq. (34) for $\sigma$,

$$\sigma = \frac{1}{2 \pi} \cosh^{-1} \left[ \pm \left( 1 + \frac{q^2}{1 - K^2 / D^2} \right) \cosh 2 \pi D \sqrt{1 - K^2 / D^2} \right] - D, \quad (40)$$

where $q = (b / g)(1 / 4 \pi)$. In Fig. 6, we show the instability growth rate for $\sigma = 0.6$, $\kappa_c = 0.3$, and $b / g = 0$, 2.4, and 4.5. $b / g = 0$ is the reference case corresponding to the nonoscillating ablation front. For $b / g = 2.4$, we are below the upper limit (see Fig. 7) and the dynamic cut-off wave number is reduced to $\kappa_m = 0.15$, while the maximum growth rate is reduced from $\sigma_{\max} = 0.112$ to $\sigma_{\max} = 0.096$ [curve (b)]. When $b / g = 4.5$ (Fig. 7), we are below the lower limit if $\kappa \leq \kappa_m$, and, thus, the maximum growth rate in this region results to $\sigma_{\max} = 0.075$ [curve (c_)], less than the previous case. However, for $\kappa_m = \kappa \leq \kappa_m$, we are in the region of parametric instabilities and, in such a region, not only the range of unstable wave numbers is wider ($\Delta \kappa_m = \kappa_m - \kappa_m$) but also the maximum growth rate has increased to a value close to the nominal case: $\sigma_{\max} = 0.104$ [curve (c+)]. Therefore, the better strategy is to remain inside the stability region defined by the parametric instabilities in the upper boundary and by the minimum wave number $\kappa_m$ above which we want to stabilize the ablation front in the lower boundary.

1. Compressibility effects

Figures 4–7 show that relatively large values of $b / g$ are required in order to be inside the stability region. This conclusion is in agreement with the experimental and theoretical results found for the case of Newtonian fluids. It would be impossible to generate such large oscillating accelerations if the unablated material ahead the ablation front were perfectly incompressible. As it was noticed by Boris and Betti, dynamic stabilization in an ablation front can work because the layers of the unablated material close to the ablation surface are compressed and decompressed by the oscillating pressure. Thus, it generates local accelerations $b$ that can be considerably larger than the background acceleration $g$ affecting the whole mass of the accelerated shell. More precisely, these compressibility effects make the generation of an oscillating acceleration in the region close to the ablation front possible. The ablation pressure itself can push the ablation front but is unable to pull it. Thus, this latter part of the oscillating cycle depends on the decompression of the layer ahead the front.

It is outside of the scope of the present work to give a detailed account of the process for which large values of $b / g$ can be achieved in an ablation front as well as to determine the thickness of the region affected by the vibration. Such a task could be conveniently performed by means of one-dimensional numerical simulations. Nevertheless, we can consider such compressibility effects in a phenomenological
manner in order to understand what new constraints are imposed on the stability region found in Sec. III A.

On the one hand, note that the minimum wave number \( k_m \) above which the front is dynamically stabilized cannot be arbitrary chosen since it will depend on the oscillation frequency.\(^7\) In fact, the thickness of the shell region that will be affected by the oscillation is the one that can be reached by the sound waves, traveling with velocity \( c_{s2} \), during an oscillation characteristic time of the order of \( \omega^{-1} \). That is,

\[
k_m^{-1} = \Delta y \approx c_{s2} \omega^{-1}.
\]

(41)

Using the dimensionless variables defined in Eq. (24), we get

\[
\kappa_m = M_2 \sigma,
\]

(42)

where \( M_2 = v_2 / c_{s2} \) is the Mach number in the relatively dense and cold material ahead the ablation front \((y < 0)\). This equation shows that the larger is the frequency \( \sigma \), the larger will be the minimum value of the of \( \kappa_m \), suggesting that too large frequencies may not stabilize sufficiently small wave numbers.

On the other hand, large frequencies are necessary to produce large local accelerations \( b \). In fact, the amplitude \( b \) of the acceleration modulation, produced by an oscillation of amplitude \( \Delta p \) in the ablation pressure, can be estimated as \( b \approx \Delta p / \rho g \Delta y \).\(^7\) Since the ablation pressure is \( p \approx \rho g d \), where \( d \) is the shell thickness, and introducing dimensionless variables, we get

\[
b \approx \frac{\Delta p \sigma}{\rho M_2^2}.
\]

(43)

Taking into account that \( \Delta p / p < 1 \), it turns out that relatively high frequencies \((\sigma \gg M_2)\) are required to produce values of \( b / g \) considerably larger than unity. So, the opposite requirements of Eqs. (42) and (46) indicate that we should use the lowest possible frequency that allows for entering in the stability region.

The consequences of the two previous conditions on the boundaries of the stability region can be seen by introducing the frequency dependent value of \( \kappa_m \) given by Eq. (42) into the negative branch of Eq. (34), with \( \sigma = 0 \), in order to calculate the lower limit for a fixed value of the Mach number \( M_2 \). Once this is done, the region of the stability chart that can actually be achieved will be determined by the minimum between the values given by Eq. (43) and by the positive branch of Eq. (34). A typical case is shown in Fig. 8 for a reference case with \( \kappa_c = 0.3 \). One can see that if the Mach number of the dense shell is known, a unique value of the minimum wave number \( \kappa_m \) that can be stabilized is determined, as well as the value of \( b / g \) and of the oscillation frequency \( \sigma \) that must be used. For instance, if \( M_2 = 0.1 \), we will be able to stabilize the wave numbers larger than \( \kappa_m = 0.08 \), and for this, it will be necessary to achieve an acceleration modulation \( b / g \approx 4.54 \) and a frequency \( \sigma \approx 0.8 \). For a sufficiently small Mach number \( M_2 \), depending on the value of \( \Delta p / p \) in the ablation pressure modulation, Eq. (43) does not impose any new constraint to the upper boundary.

\[
\beta = 2 \sqrt{R^2 - D^2} \cos 2 \pi \sqrt{R^2 - D^2} \cos 2 \pi \eta \sin 2 \pi \sqrt{R^2 - D^2}.
\]

(44)

By introducing the dimensionless magnitudes defined in Eq. (24) and taking Eq. (33) into account, this equation can be written as follows:

\[
\Delta p / \rho = 1
\]

FIG. 8. Stability region for \( \kappa_c = 0.3 \). The lower boundaries (“−”) are given for \( \kappa_c = M_2 \sigma \), and different values of the Mach number \( M_2 = 0.04 \), \( 0.06 \), \( 0.10 \), and \( 0.20 \). The upper boundaries (“+”) correspond to the incompressible case of Fig. 3 for the smallest values of \( M_2 \) and to a pressure modulation of a 100% for the case \( M_2 = 0.20 \).

In any case, the minimum value of \( \kappa_m \) is rather insensitive to the exact value of \( M_2 \) but, instead, it considerably affects the required values of \( b / g \) and \( \sigma \).

B. Asymmetric driving

There are several types of driving involving sequences of Dirac deltas and/or square waves that can be considered for the analysis of dynamic stabilization of RTI and that are analytically tractable, at least in a partial degree. They could be used to study the effect of the shape of the modulation on the stability charts in order to find the optimum kind of driving. Such a task is beyond the purpose of the present work; however, the analysis presented so far allows for the consideration of the simplest asymmetric driving. It consists of an impulsive driving composed of negative (subtracting from the background gravity) Dirac deltas and it can be compared with the previous symmetric case. Such a kind of modulation results from Eq. (21) by taking \( \beta_1 = -\beta_2 = 0 \). Then, the dispersion relation can be obtained as before by solving Eq. (28) in the intervals when \( \Gamma(\tau) = 0 \) and by imposing the matching conditions at \( \tau = 2m \pi \). In this manner, it is rather easy to get the following dispersion relation:

\[
\beta = 2 \sqrt{R^2 - D^2} \cos 2 \pi \sqrt{R^2 - D^2} \cos 2 \pi \eta \sin 2 \pi \sqrt{R^2 - D^2}.
\]
\[ b/g = 4\sigma \sqrt{1 - \frac{K^2 \cosh 2\pi D(1 - K^2/D^2) \pm \cosh 2\pi (\sigma + D)}{D^2 \sinh 2\pi D(1 - K^2/D^2)}}, \]

where \( K \) and \( D \) are given by Eq. (35). As in the previous case, the limits of marginal instability correspond to the case \( \sigma = 0 \). In Fig. 9, we present such limits for the particular cases \( \kappa_c = 0.3, \sigma = 0.8 \) and \( \kappa_c = 1, \sigma = 0.5 \) that have also been considered in Fig. 2 for the symmetric modulation. The main qualitative difference between both cases is that now the lower limit \( b/g \) remains finite for small wave numbers,

\[ \frac{b}{g} \bigg|_{-} = 2\pi \quad (\kappa \ll 1). \]

Instead, the upper limit for \( \kappa \ll 1 \) is the same as given by Eq. (37). On the other hand, for \( \kappa \gg 1 \), both the upper and the lower limits grow exponentially in a somehow stronger way than in the symmetric case,

\[ \frac{b}{g} \bigg|_{\pm} = 4\sigma \left[ 1 \pm e^{(\pi/2\sigma)^2[(\kappa/\kappa_c)^{2/3} - 1]} \right] \quad (\kappa \gg 1). \]

These differences cause the minimum value of the upper limit to be somehow higher than in the symmetric case and the minimum value of the lower limit to be restricted to values \( b/g \leq 2\pi \). Such a behavior would allow, in principle, dynamic stabilization of all perturbation wave numbers provided that the compressibility effects are not taken into account and that the driving frequency is high enough to make the upper limit larger than \( 2\pi \). Of course, since the compressibility effects are essential to produce values of \( b/g \gg 1 \), such a possibility will not really exist and we will only stabilize wave numbers larger than a certain value \( \kappa_m \). Assuming a fixed value of \( \kappa_m \), we can find the stability region in the plane \((b/g, \sigma)\) for a given value of the cut-off wave number \( \kappa_c \). In Fig. 10, we show the results for \( \kappa_c = 0.3 \) and the same three values of \( \kappa_m \) (\( \kappa_m = 0.1, 0.15, \) and 0.3) considered in Fig. 3 so that they can be compared. The resulting stability region is wider and it can be achieved with relatively lower values of \( \sigma \) and \( b/g \). In Fig. 11, for \( \kappa_c = 1 \) and assuming that compressibility effects are not present, we can see that, for \( b/g \geq 2\pi \), all wave numbers could be stabilized.

FIG. 9. Asymmetric impulsive driving. Dimensionless driving acceleration \( b/g \) for marginal stability as functions of the dimensionless wave number \( \kappa \) for \( \kappa_c = 0.3, \sigma = 0.8 \) and for \( \kappa_c = 1, \sigma = 0.5 \). Symbols “+” and “−” denote the stability upper and lower limits, respectively.

FIG. 10. Asymmetric impulsive driving. Stability region for \( \kappa_c = 0.3 \). The lower boundary is given for \( \kappa_m = 0.10, 0.15, \) and 0.20.

FIG. 11. Asymmetric impulsive driving. Stability region for \( \kappa_c = 1 \). The lower boundary is given for \( \kappa_m = 0.0, 0.1, \) and 0.3.
Also for this case, it turns out that the dynamic cut-off wave number is reduced, for given values of $b/g$ and $\bar{\sigma}$, in comparison with the symmetric driving.

In addition, the growth rate $\gamma v^2/g = \sigma \bar{\sigma}$ can be easily obtained by solving Eq. \((45)\) for $\sigma$,

$$\sigma = \frac{1}{2\pi} \cosh^{-1} \left[ \frac{1}{\cosh 2\pi \sqrt{1 - K^2/D^2}} \right] - D. \quad (48)$$

The maximum growth rate is reduced, as it can be seen by comparing Figs. 12 and 13 with the corresponding Figs. 6 and 7 for the symmetric case. These relative advantages of the asymmetric driving still hold when the compressibility effects that allow for values of $b/g > 1$ are included. This is shown in Fig. 14, which corresponds to the same parameters used in Fig. 8 so that both cases can be compared ($\kappa_c = 0.3$; $M_2 = 0.04, 0.06, 0.1, 0.2$). It is possible to get lower values of the minimum wave number $\kappa_m$ with relatively lower values of $b/g$ and $\bar{\sigma}$ for each given value of $M_2$.

It is beyond the scope here to look for the optimum wave form of the driving that minimize both $\kappa_m$ and the maximum growth rate, and although the results for the asymmetric impulsive driving are clearly more satisfactory than the ones corresponding to the symmetric case, such an asymmetric driving is not of practical interest by itself. In fact, such an asymmetric modulation would lead to the continuous decomposition of the shell region of thickness $\Delta y = c_2 \sigma^{-1}$ and the only effect of the modulation would be to translate the unstable interface to the position $y \sim -k_m^{-1}$. In fact, any type of driving should maintain the density average value $\rho_o$ of the dense layer ahead the front in such a way that at the end of each cycle, the interface should go back to its equilibrium position. However, the previous results are useful for calculating the growth rate of the following asymmetric modulation that satisfy the requirement of preserving the value $\rho_o$ of the average shell density:

![Diagram](image-url)
rather robust method for mitigating RTI in an ablation front. Nevertheless, further analysis would be necessary in order to find the optimum driving.

IV. CONCLUDING REMARKS

We have theoretically analyzed the problem of the dynamic stabilization of RTI in an ablation front by using the simplest type of acceleration driving consisting of sequences of Dirac deltas. Such kind of modulations in the acceleration allow for finding explicit analytical solutions for the instability dispersion relation that also yield analytical expressions for both the instability growth rate and the boundaries of the region of marginal stability.

We demonstrate that as in the case of RTI in Newtonian fluids, the presence of damping effects is essential to dynamically stabilize all wave numbers above some minimum $k_m$. However, the existence of a cut-off wave number $k_c$ in the RTI of the nonoscillating ablation front is not required, and dynamic stabilization turns out to be possible even when $k_c=\infty$. This behavior is different from the one observed in Newtonian fluids and is of special significance for the case of ablation fronts directly driven by ion beams where the practically absence of beam energy transport by thermal diffusion makes $k_c$ very large. Nevertheless, the present work shows that the requisites for dynamic stabilization become more exigent as the minimum value $k_m$ of the wave number gets smaller, above which we want to stabilize the front.

Furthermore, we have considered the constraints imposed on the stability region by the compressibility of the fluid layers ahead the front in a phenomenological manner. Such effects preclude the possibility of freely choosing the value of $k_m$ since it is determined by the oscillation frequency $\omega$. This frequency, on the other hand, cannot be arbitrarily small because relatively high values are necessary for generating high local values of $b/g$. Therefore, a minimum frequency is required to reach the stability region and thus determines the minimum value of $k_m$.

Analysis of the effect of different types of driving show that more study is needed to find the optimum acceleration driving. However, we have also presented the results of asymmetric impulsive driving, showing that the conclusions extracted from the symmetric impulsive driving also hold for the asymmetric case. Nonetheless, significant quantitative differences are found, indicating that results can be improved by using an appropriate wave form for the acceleration modulation.

Asymmetric impulsive driving itself is not of practical interest for dynamic stabilization since it leads to a continuous decompression of the layer subjected to vibration. However, such results can be easy extended for calculating the growth rate of an equivalent problem in which the interface returns to its initial relative position at the end of each cycle, ensuring that the average value of the density of the vibrated layer is maintained. We find that such a kind of driving yields worse results than the symmetric impulsive driving but both of them significantly reduce the maximum growth rate as well as the dynamic cut-off wave number $k_m$. Finally,
we also show that a symmetric square wave yields even better results.

We cannot say if further improvements could be achieved by carefully tuning the wave form of the acceleration modulation, but the present study suggest that combinations of Dirac deltas and square waves, which are analytically tractable, can be used as a guide for an optimization study. In addition, a numerical simulation analysis would also be required to address the role of compressibility in a quantitative manner.

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4 L. J. Perkins (private communication), 2009.