Rayleigh–Taylor instability of steady ablation fronts: The discontinuity model revisited

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A new model for the instability of a steady ablation front based on the sharp boundary approximation is presented. It is shown that a self-consistent dispersion relation can be found in terms of the density jump across the front. This is an unknown parameter that depends on the structure of the front and its determination requires the prescription of a characteristic length inherent to the instability process. With an adequate choice of such a length, the model yields results, in excellent agreement with the numerical calculations and with the sophisticated self-consistent models recently reported in the literature. © 1997 American Institute of Physics.

I. INTRODUCTION

One of the main factors that determines the minimum energy required for the ignition of an inertial confinement fusion target is the onset of Rayleigh–Taylor instabilities during the implosion phase.1 For this reason such an instability is a current subject of extensive experimental,2–4 simulational,5–8 and theoretical research.9–24 In particular, the theoretical studies have followed three essentially different ways of approaching the problem. First, we mention the completely self-consistent studies based on the numerical solution of the fluid equations as an eigenvalue problem.12,13,15 The advantage of this kind of analysis over the simulations is that the equilibrium situation is clearly defined in the simplest way. Specifically, it consists of a stationary ablative corona driven by electronic thermal conduction or a more general diffusive process of energy transport. These calculations provide quantitative results, but its numerical nature still makes it difficult to reach a complete understanding of the underlying physics.

Very recently, self-consistent analytical or semianalytical models have been developed.17–22 Such models apply elaborate mathematical tools (perturbation theory and asymptotic matching) and allow us to obtain solutions in good agreement with the numerical works previously mentioned.

Finally, we have the simplest treatments founded on the sharp boundary model (SBM).9–11,14,18,24 According to this model, the ablation front can be taken as a moving surface of zero thickness, which is initially at \( y = 0 \) and separates two homogeneous fluids of densities \( \rho_1 \), for \( y > 0 \), and \( \rho_2 > \rho_1 \), for \( y < 0 \), respectively. The heavy fluid is supported against an acceleration \( g \) by the lighter one (Fig. 1). We want to remark that the paper by Sanz23 also deals with a kind of sharp boundary model that considers an interface separating two fluids. But these fluids are not homogeneous and the model takes into account the structure on both sides of the front in a self-consistent way. Here we reserve the term “sharp boundary model” for the simplest situation in which the interface separates two homogeneous fluids. The main shortcoming of this model is that it requires additional information associated with the flow structure behind the ablation front. Such information cannot be introduced self-consistently with the SBM and it has been usually considered to be the reason for its failure in reproducing the results of the numerical calculations. As it was originally noted by Bodner,9 we have to deal with the corresponding problem arising when the ablation process itself (or, in general, a weak expansion) is studied by means of the discontinuity approximation.25 It is well known, however, that the main properties of an ablative corona can be correctly described with this model if the characteristic length of the energy deposition mechanism driving the ablation is adequately introduced (see Refs. 26–28 and references cited therein). We would expect a similar situation in the study of the front stability and, then, a proper consideration of the physical process occurring at the ablation front should lead us to a satisfactory model for the Rayleigh–Taylor instability based on the SBM. It is clear that the availability of such a simple model would be of great help for improving our understanding of the forces driving the instability and the mechanisms that control the ablative stabilization. Therefore, let us reexamine the SBM in light of the new results yielded by the self-consistent models recently reported, in order to identify the most essential physics. Of course, a noncomplete self-consistency is the price we have to pay for obtaining a simple picture, but instead, we may get a new tool for studying instability regimes of potential interest (see Refs. 21, 29, and Sec. III).

With this purpose in mind, here we briefly discuss the nature of the inconsistency in the SBM. As it is well known, it lies in the fact that the model does not contain information regarding the profiles of density, pressure, and velocity created as a consequence of the ablation and, as we have already mentioned, we run into a similar problem when the discontinuity model is used in the study of the ablation process itself. Then, it may be instructive to consider such a case as...
an example. A schematic representation of the typical SBM for an ablation front is shown in Fig. 1, where \( Q_0 \) is the energy flux driving the ablation and \( \rho_2 \) is the density of the dense phase. Both are, in principle, known physical parameters. In addition, \( \rho_1 \) is the characteristic density of the ablated material, and \( v_2 \) and \( v_1 \) are, respectively, the fluid velocities ahead and behind the front, and \( \varepsilon \) is the specific internal energy. With the usual approximations for a weak expansion, the conservation equations of mass, momentum, and energy read as

\[
\begin{align*}
\dot{m} &= \rho_2 v_2 = \rho_1 v_1, \\
p_a &= \rho_1 v_1 = \rho_2 v_2, \\
p_1 v_1^3 &= Q_0,
\end{align*}
\]

where \( \dot{m} \) is the ablation rate, \( p_a \) is the ablation pressure, and we are interested in calculating these magnitudes in terms of the known parameters \( Q_0 \) and \( \rho_2 \). After some straightforward algebra, we obtain

\[
\dot{m} = r_d^{2/5}(Q_0 \rho_2^{2/5})^{1/3}, \quad p_a \approx r_d^{1/3}(Q_0 \rho_2^{1/5})^{1/3},
\]

where \( r_d = \rho_1/\rho_2 \) is an unknown parameter that must be specified in order to close the problem or, equivalently, we need to give the characteristic scale \( L_d \) of the gradients in the corona. Such a scale is related to that characterizing the particular process of energy deposition \( L_Q \), by means of the so-called “self-regulating” hypothesis. In order to fix ideas, let us consider, for instance, the ablation of a spherical target of radius \( r_0 \) driven by electronic thermal conduction. In such a case, we have \( L_d \approx r_0 \), \( Q_0 \approx \chi T^{1/2} \nabla T \), and \( L_Q \approx \chi v_1^4/\rho_1 \) (\( \chi \) is the coefficient of thermal conduction and \( T \) is the characteristic corona temperature). Then, according to the self-regulating condition, it must be \( r_0 \approx \chi v_1^4/\rho_1 \), and \( \rho_1 \) can be determined in terms of \( Q_0 \), \( p_2 \), and \( r_0 \). Within the context of the SBM, the self-regulating condition is an external prescription that cannot be derived from the conservation equations and makes the resulting expressions for \( \dot{m} \) and \( p_a \) not self-consistent. However, this is certainly a correct hypothesis that finds justification once the problem is solved by integrating the complete fluid equations. In doing so, we see that it stems from the boundary conditions, that is, from the imposition that the temperature must go to zero (or to some specific value) on the ablation surface.

Two conclusions can be extracted from the previous example: first, it is possible to obtain adequate results with the discontinuity model, even if they are not self-consistent, provided that the correct extra information is incorporated, either by doing a good physical guess or by obtaining it from a complete solution. Second, we can write self-consistent “solutions” in terms of the unknown parameter \( r_d \).

It is a matter of fact that previous treatments of the Rayleigh–Taylor instability by means of the SBM have not produced satisfactory results. But it is also true that those models have used drastic approximations such as adiabatic perturbations on both sides of the interface and/or negligible thermal flux across the front. These assumptions have usually led to internal inconsistencies, independently of the particular choice of the extra boundary condition. In these circumstances, it is clear that satisfactory solutions cannot be expected, even if we suppose that a correct condition is used to close the problem.

In this paper we present a new version of the sharp boundary model in which the thermal flux \( Q \) is considered across and behind the front, and its perturbation \( \partial Q \) is written in terms of the unperturbed value \( Q_0 \) by exploiting the fact that the isotherms move with the front. Thus, we are able to derive a self-consistent dispersion relation in terms of an unknown parameter. Similarly to the problem of the ablative expansion discussed above, this unknown parameter is a self-consistent density jump associated with the instability process. The dispersion relation in terms of the density jump \( r_d = \rho_1/\rho_2 \) is derived in Sec. II from a linear stability analysis. In Sec. III, we calculate \( r_d \) by using a simple corona model and by prescribing the characteristic length relevant to the instability problem, in agreement with the complete solutions given in Refs. 19–23. Besides, we extend the model to the interesting regime in which lateral transport becomes a relevant contribution to the gradient effects. The mechanism of stabilization is discussed in Sec. IV and an alternative way to derive the dispersion relation is shown. The main conclusions are summarized in Sec. V.

II. LINEAR ANALYSIS IN TERMS OF NORMAL MODES

We consider a steady ablation front as a surface of zero thickness placed at \( y = \xi(x,t) \). This surface separates two fluids that are initially homogeneous and they have densities \( \rho_2 \) and \( \rho_1 \) (\( \rho_2 > \rho_1 \)), respectively, ahead of and behind the front (Fig. 2). The acceleration \( g \) is opposite to the density gradient and it is taken in the direction of the positive y axis. As it is known, such a model can be a good representation of a realistic ablation front, provided that the characteristic length \( L_D = \min(|\partial \rho/\partial y|) \) is much smaller than \( k^{-1} \) (k is the perturbation wave number and \( k L_D \leq 1 \)). Thus, we perform the stability analysis in the usual manner by linearizing the fluid equations:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,
\]
\[ (\gamma + q v_0) \rho_0 \delta v_p + q \delta p = 0, \]  
\[ (\gamma + q v_0) (\delta p - c_0^2 \delta p) = (q^2 - k^2) \frac{\kappa_{\rho_0}}{\rho_0} \left( \delta p - \frac{3}{5} c_0^2 \delta p \right). \]

Notice that the energy equation [Eq. (11)] contains an explicit dependence on the thermal conductivity \( \kappa_{\rho_0} \). For simplicity, we will assume that the effect of the thermal transport can be neglected ahead of the front \((y < 0)\), but that it is relatively strong behind it:

\[ \frac{\kappa_{D_2}}{\rho_2 v_2} \ll 1 \quad \text{for } y < 0 \quad \text{(adiabatic)}, \]
\[ \frac{\kappa_{D_1}}{\rho_1 v_1} \ll 1, \quad \text{for } y > 0 \quad \text{(isothermal)}. \]

Therefore, we obtain the following characteristic equations, giving the possible perturbation modes in the regions ahead of and behind the front, respectively:

\[ (\gamma + q v_2)^2 [(\gamma + q v_2)^2 + (q^2 - k^2) c_2^2] = 0 \quad (y < 0), \]
\[ (\gamma + q v_1)^3 [(\gamma + q v_1) - (q^2 - k^2) \frac{3 \kappa_{D_1}}{5 \rho_1}] \]
\[ \left[ - (\gamma + q v_1) (q^2 - k^2) \left( (\gamma + q v_1) - (q^2 - k^2) \frac{3 \kappa_{D_1}}{5 \rho_1} \right) = 0 \right. \]
\[ (y > 0). \]

Then, taking into account Eqs. (7), (12), and (13), we find the possible longitudinal wave numbers from the roots of Eqs. (14) and (15). Ahead of the front we obtain two sonic modes; one entropy mode, and one vorticity mode:

\[ q_{2s} = \pm k, \quad q_{2s,v} = - \frac{\gamma}{v_2}. \]

and behind the front Eq. (15) yields two sonic modes, two thermal conduction modes, and one vorticity mode:

\[ q_{1s} = \pm k, \quad q_{1s,v} = \pm k, \quad q_{1v} = - \frac{\gamma}{v_1}. \]

It may be of interest to note that the thermal conduction modes obtained in Ref. 14 turn into the present ones in the limits given by Eqs. (7) and (13).

In order to find the possible solutions we have to impose proper boundary conditions. Since the instability is expected to be localized at the interface, the perturbations have to vanish at \( y \to \pm \infty \). So, for the income flow \((y < 0)\) only one solution vanishing to infinity (a sonic mode) is found:

\[ \delta v_{2s} = - l \quad \delta v_{2s} = A e^{ky}. \]
\[ \delta p_2 = - \frac{\rho_2}{k} (\gamma + q v_2) \delta v_{2s} = c_2^2 \delta p_2. \]

Behind the front \((y > 0)\) there are three possible solutions that decay at \( y \to + \infty \). We have the sonic mode \((q_{1s} = - k)\):
\begin{align}
\delta v_{1,yy} &= i \delta v_{1,xx} = B e^{-ky}, \\
\delta p_{1x} &= \frac{\rho_1}{k} (\gamma - kv_1) \delta v_{1,yy} = c_1^2 \delta \rho_{1y}; \\
\delta v_{1,yy} &= i \delta v_{1,xx} = C e^{-ky}, \\
\delta \rho_{1y} &= \frac{\rho_1}{k} (\gamma - kv_1) \delta v_{1,yy} = -\frac{\kappa_{D1} k}{\rho_1 v_1} M_1^2 c_1^2 \delta \rho_{1T} \\
&[\text{where } (\kappa_{D1} k / \rho_1 v_1) M_1^2 \ll 1, \text{ and we have used that } \gamma \ll kv_1;] \\
\delta \rho_{1T} &= 0, \quad \delta \rho_{1y} = 0, \\
\delta v_{1,y} &= \frac{1}{k} v_1 \delta v_{1,xx} = D e^{-\gamma y/v_1}, \\
\phi &= \phi_0 (\xi) - \frac{d \phi_0}{dy} \bigg|_{y=0} + \delta \phi. \\
\end{align}

Introducing this expression into the conservation equations and integrating across the front surface, we find the following jump conditions in the limit \( k L_D \ll 1 \):\(^{9-11}\)

\begin{align}
\Delta [-\rho_0 \gamma \xi_f + \rho_0 \delta v_y + v_0 \delta \rho] &= 0, \\
\Delta [\delta v_x + i k \xi_f \delta v_0] &= 0, \\
\Delta [\delta \rho + \rho_0^2 \delta \rho + 2 \rho_0 v_0 \delta v_y + \rho_0 g \xi_f] &= 0, \\
\Delta \left[ -\left( \rho_0 \frac{v_0^2}{2} + \frac{3}{2} \rho_0 \right) \gamma \xi_f + \frac{3}{2} \rho_0 v_0 \delta v_y + \frac{v_0^3}{2} \delta \rho \right. \\
&\left. + \frac{5}{2} \rho_0 \delta v_y + \frac{5}{2} v_0 \delta \rho + \delta Q_y ] = 0, \right)
\end{align}

\begin{align}
\Delta [\delta Q_y] &= -\delta Q_{1,y} \approx q \kappa_{D1} \delta \epsilon_1,
\end{align}

where \( \Delta [\varphi] = \varphi_i - \varphi_f \) denotes the jump of a quantity \( \varphi \) across the interface. In writing Eq. (27) we have used Eqs. (12) and (13). Besides, we have taken into account, consistently with the present thin front approximation, that the isotherms move with the front.\(^{9-11,17,19,23}\) The last fact has been originally noted by Bodner\(^a\) and Baker,\(^b\) and it turns out in a rigorous way from the self-consistent theory by Sanz.\(^{19,23}\) Such a property of the front arises from the fact that the maximum deposition rate of the energy flux takes place on the ablation surface, independently of the displacement produced by the perturbation, and, as a consequence, the flux across the front and the front temperature are not affected by this displacement. Thus, when a fluid element on the ablation front with unperturbed temperature \( \epsilon_0 (y=0) \) is displaced from \( y=0 \) to \( y=\zeta \) as a result of the perturbation, its temperature does not change, and so the perturbed temperature is

\begin{equation}
\epsilon (\zeta) = \epsilon_0 (0),
\end{equation}

and the perturbation of the temperature is\(^{17,23}\)

\begin{equation}
\delta \epsilon = -\xi \frac{d \epsilon_0}{dy} \bigg|_{y=0}.
\end{equation}

We can exploit this property further for writing Eq. (27) in a more suitable form. Indeed, Eq. (27) has a limited practical application, as it requires the knowledge of the local value of the thermal conductivity \( \kappa_T \). This value changes drastically through the interface [see Eq. (12) and (13)] in a manner that strongly depends on the details of the ablation front structure. Instead, we can use Eq. (29) to express \( \delta Q_y \) in terms of the unperturbed thermal flux \( Q_0 \), which relates to the global properties of the corona and is considered a known parameter (in terms of the ratio \( \rho_1/\rho_2 \)):\(^{9,10}\)

\begin{equation}
\Delta [\delta Q_y] = -\delta Q_{1,y} \approx k \xi Q_0, \quad Q_0 = \kappa_{D1} \frac{d \epsilon_0}{dy} \bigg|_{y=0}.
\end{equation}

where we have taken \( q = -k \) and considered Eqs. (12) and (13). Additional progress can be made by using the zeroth-order energy equation for the flow behind the front. In the corona region \( (y>0) \) and for \( M_1^2 \ll 1 \) we have\(^{14,18-23}\)

\begin{equation}
Q_0 = \frac{5}{3} \dot{m_0} (\epsilon_1 - \epsilon_2) = \frac{5}{2} \rho_1 v_1 (1 - r_D), \quad r_D = \frac{\rho_1}{\rho_2}.
\end{equation}

With these considerations, we evaluate the perturbations on each side of the interface by means of Eqs. (18)–(21) with the boundary conditions given by Eqs. (23)–(26), and we obtain the following equations:

\begin{align}
A &= (\gamma + kv_2) \xi_f, \\
A + (B + C) + \frac{\gamma}{k v_1} D &= kv_1 \xi_f (1 - r_D), \\
\rho_2 \frac{(\gamma + kv_2) A + \rho_1 (\gamma - kv_1) (B + C)}{k} &= (\rho_2 - \rho_1) g \xi_f \\
-2 \rho_0 v_0 k v_1 \xi_f (1 - r_D), \\
A - (B + C) - D &= -kv_1 \xi_f (1 - r_D),
\end{align}

where Eqs. (7) and (13) have been used. As we expected, we have more unknowns than equations but note, however, that the sonic and the thermal modes behind the front \( (B \text{ and } C) \) appear coupled as a consequence of the fact that the interface representing the front is considered to be an isotherm.

In fact, such a property of the ablation front allows us to calculate \( B \) and \( C \) from Eqs. (27) to (31):

\begin{equation}
\frac{4 \kappa_{D1}}{5 \rho_1} (\gamma - kv_1) \left[ B + \frac{3}{2} M_1^2 k \kappa_{D1} - C \right] \approx -kv_1 \xi_f (1 - r_D),
\end{equation}

When the thermal transport behind the front is neglected, as it was a common practice in previous versions of the SBM,
there is no way to exploit this property, and, hence, one of the previous equations is missed. Here, we obtain five equations for calculating six unknowns: $A$, $B$, $C$, $D$, $\xi_f$, and $r_D$, and it allows us to obtain a self-consistent dispersion relation in terms of the density jump $r_D$:

$$\gamma^2 + \frac{4k\nu_2}{1+r_D} \gamma + k^2 \left[ \frac{\nu_2^2}{g r_D} - A_T \right] = 0, \quad A_T = \frac{1-r_D}{1+r_D}, \quad (37)$$

and the resulting growth rate is

$$\gamma = \sqrt{\left( \frac{2k\nu_2}{1+r_D} \right)^2 - k^2 \left[ \frac{\nu_2^2}{g r_D} - A_T \right]} - \frac{2k\nu_2}{1+r_D}. \quad (38)$$

This formula agrees with the results of Refs. 19 and 23 and is formally identical to that obtained in Refs. 20 and 21 provided that a convenient characteristic length associated with the instability process is prescribed in order to calculate $r_D$.

According to the results of Refs. 19–23, excellent agreement with the Kull numerical results is achieved by taking:

$$r_D \approx \left( \frac{2kL_0}{\nu} \right)^{1/\nu}, \quad L_0 = \frac{3\kappa_2}{5\rho_2\nu^2}, \quad \kappa_2 = \chi e_2^\nu. \quad (39)$$

As we will see later from a simple corona model, this expression is consistent with the choice that the required characteristic length is of the order of $k^{-1}$. Extensive comparisons with numerical results have been performed in the above-mentioned references and they show the validity of the previous formulas [Eqs. (38) and (30)]. Therefore, hereafter we will concentrate on the discussion and the extension of these results.

III. DENSITY JUMP CALCULATION AND THE EFFECT OF LATERAL TRANSPORT

As we have previously mentioned, there is no way within the context of the SBM to determine $r_D$ in a self-consistent manner, and hence, we have to introduce a physical assumption for its estimation. If we identify the cold phase density $\rho_2$ with the maximum density in the corona, we can find $r_D$ by calculating the density $\rho_1 = \rho(\gamma = \gamma^*)$ with a simple corona model in which we take $\gamma^*$ according to the results of the self-consistent models. Since we are dealing with surface modes that decay from the ablation front as $\exp(-k\gamma)$ [see Eqs. (19) and (20)], we can expect that only the corona region within a distance $\gamma^* \sim k^{-1}$ will be involved in the instability development. Besides, we notice that the other two characteristic lengths entering in the problem are much larger or much smaller than $k^{-1}$ [see Eqs. (12) and (13)]. Thus, the characteristic length to be used for calculating the density jump must be of the order of $k^{-1}$, which relates to the two-dimensional effects introduced by the instability. Excellent agreement between the present model and the self-consistent theory and the numerical calculations is achieved by taking $\gamma^* = (2k)^{-1}$. On the other hand, the following approximate expressions for the temperature and density profiles can be obtained from a corona model:

$$\frac{\epsilon}{\epsilon_2} \approx \frac{\rho_2}{\rho}, \quad \left\{ \begin{array}{l} (\nu_2 L_0)^{1/\nu}, \quad y \gg y_0, \\
(1 + a e^{y/y_0}), \quad y \ll y_0, \end{array} \right. \quad (40)$$

where

\begin{align*}
\psi &= \frac{v}{\nu_0 L_0} \quad (1-k^2
\end{align*}
cannot be expected to be valid in the limit \( G \) as, strictly, the WKB theory has been derived. In fact, in mechanism arising as the WKB theory, that lateral transport is the new physical \( kL_0 \) extending the sharp boundary model to the regime in which \( r \) is of the order of unity and by assuming, on the basis of the WKB theory, that lateral transport is the new physical \( kL_0 \) extending the sharp boundary model to the regime in which

\[
\Delta[\delta Q_x] = -ik\int_0^{\gamma_L - L_0} \delta Q_x \, dy \approx \phi_0 k^2 \kappa_0 \delta \varepsilon; \quad \phi_0 = 1 + kL_0, \tag{44}
\]

where we have used Eqs. (29), (31), and (40). It is interesting to note that the contribution to the energy flux perturbation arising from the lateral conduction is \( k^2 \kappa_0 L_0 \delta \varepsilon \) and that it was the only term considered in Ref. 10 in studying the effect of the thermal conduction by means of the SBM. Using Eq. (44) into Eq. (26), we can find the modified growth rate that includes the lateral transport:

\[
\gamma = \sqrt{\frac{(1 + \phi_0) k v_2}{1 + r_D}} - k g\frac{2 \phi_0 - 1 + r_D}{1 + r_D} \frac{k v_2^2}{g r_D} A f, \tag{45}
\]

where \( r_D \) is evaluated as before, by means of Eq. (40). We have represented the cutoff wave number given by this equation in Fig. 3 (solid line) and, as can be seen, an excellent agreement with the numerical calculations is found for the whole range of values of the parameter \( \Gamma \). In Fig. 4, we have compared the growth rate given by Eq. (45) with the numerical results reported in Ref. 15 for \( \Gamma = 20 \) and \( \Gamma = 100 \) \((\nu = \frac{1}{2})\).

In every case the parameter \( r_D \) has been calculated as above, using Eqs. (40)–(42) with \( \gamma = (2k)^{-1} \).

It is worth remarking that Eq. (45) has been obtained by extending the sharp boundary model to the regime in which \( kL_0 \) is of the order of unity and by assuming, on the basis of the WKB theory, that lateral transport is the new physical mechanism arising as \( kL_0 \) increases. Nevertheless, Eq. (45) cannot be expected to be valid in the limit \( kL_0 \gg 1 \), for which, strictly, the WKB theory has been derived. In fact, in such a limit the cutoff wave number given by Eq. (45) scales as \( \Gamma^{-1/2} \) instead of the scaling as \( \Gamma^{-2/5} \), predicted by the WKB theory. However, the intermediate regime considered by the previous formula is probably the most interesting to inertial fusion applications, as the extremely large values of \( \Gamma \) for which the WKB theory applies may be, in practice, very difficult to reach.

The results presented in Figs. 3 and 4, as well as those obtained in previous works,12,13,15,17,19–23 show that our simple model, the sophisticated self-consistent theory, and a few available simulations8 are in very good agreement. In all cases, a purely diffusive process of energy transport was considered. Such an agreement seems to indicate that the main physical mechanisms causing the instability of an ablation front have been reasonably understood. However, there exist some different results reported in the literature that cannot be easily included in the body of the theory previously mentioned. They are summarized in the following fitting formula:1–3,5–7

\[
\gamma = \sqrt{\frac{kg}{1 + kL_D}} - \beta k v_2, \tag{46}
\]

where \( L_D \) is the minimum scale of the density gradient defined in Sec. II and \( \beta \) is a parameter varying between 1 and 4. In the publications it is said that such results were derived from numerical simulations,1,32 but details of these calculations have not been completely reported. Therefore, it may be of importance to compare Eq. (46) with the growth rate calculated by means of Eq. (45). For this, we need to calculate \( L_D \) consistently with the corona model previously used:14,18–23

\[
L_D = \min \left( \frac{\rho}{dp/dy} \right) = \frac{(\nu + 1) \nu^{\nu - 1}}{\nu^\nu} L_0. \tag{47}
\]

In Fig. 5 we show the cutoff wave number as a function of \( \Gamma \) such as it is given by Eq. (46), for \( \beta = 1.5 \) and \( \beta = 4 \), together with the results of our model. We see that there is good agreement between the model (and the other previous results) and the fitting formula, for \( \Gamma \approx 1 \) and \( \beta = 1.5 \). The case with \( \beta = 4 \) predicts a stronger stabilization effect. It is beyond the scope of this work to explain the reasons for this discrepancy, but we can speculate, on the basis of our model, about
the possible mechanisms that could increase the stabilization effect. In the first place, it is important to observe that, for \( \Gamma > 1 \), the results of the present model are practically independent of the particular value of the exponent \( \nu \), which is characteristic of the process of diffusive transport. This is due to the fact that in such a regime the parameter \( \tau_D \) is determined by the branch \( y \leq y_0 \) in Eq. (40) and it depends weakly of \( \nu \) through Eq. (41). The same conclusion comes out from models based on the WKB approximation.

On the other hand, it is clear that the comparison presented in Fig. 5 makes sense only in the case that Eq. (46) applies to ablation driven by a purely diffusive process of energy transport. Otherwise, the characteristic length \( L_0 \) in Eqs. (40), (44), and (47), as well as the parameter \( \Gamma \), must be defined in a different way. We note that the length \( L_0 \) in Eq. (44) is much smaller than the maximum scale of the density gradient \( L_D \) [see Eq. (47)] and it corresponds to the characteristic length of deposition of the energy flux \( Q_0 \). Of course, since thermal diffusion is the only transport process considered in our model, the two lengths are proportional. But, in the case that a suprathermal component (suprathermal electrons or radiation) is also present in the corona and it has the adequate parameters for dominating the transport in the region close to the maximum density, the characteristic length in Eq. (44) may be related to the local value of the suprathermal mean-free path \( L_{ST} \). If \( L_{ST} \gg L_0 \), the stabilization effect due to lateral transport would be enhanced. In particular, the results for \( \beta = 4 \) in Fig. 5 can be well fitted by putting \( \phi_0 = 1 + 10 L_0 \) in Eq. (45). A second effect has been pointed out in Ref. 3 as coming from the energy spectrum of the radiation driving the ablation that would affect the steepness of the density profile at the ablation front. Thus, the characteristic length of the density gradient, as well as the density profile given by Eq. (40), may be completely altered.

The concordance of the several approaches, including the present simple model, in which the ablation process is well defined should indicate that adequate tools, with different degrees of complexity, are already available for the understanding of the Rayleigh–Taylor instability of a steady ablation front. However, further progress in the interpretation of sophisticated simulations may require the development of more realistic corona models than those presently available, which include a suprathermal component and/or a multi-group treatment of the energy transport process.

### IV. STABILIZATION MECHANISM

One of the advantages of the simple model presented above is that it offers the possibility of analyzing the different mechanisms contributing to the reduction of the growth rate and leading to complete stabilization for relatively small perturbation wavelengths. For this, it may be interesting to show an alternative way of obtaining the dispersion relation. As a first step, we introduce the expressions for the perturbations \( \delta m_2 \) and \( \delta \sigma_2 \), respectively, of the mass flow rate and of the momentum flux in the cold region \( (y < 0) \) relative to the moving ablation surface:

\[
\delta m_2 = \rho_2 (\delta v_{2y} - \gamma \xi ) + v_2 \delta \rho_2,
\]

\[
\delta \sigma_2 = \delta \rho_2 + 2 \rho_2 v_2 (\delta v_{2y} - \gamma \xi ) + v_2^2 \delta \rho_2
\]

\[
+ \frac{d(\rho_2 v_2^2 + p_2)}{dy}.
\]

From these equations and taking into account Eqs. (18) (with \( M_{ST} \ll 1 \)), the following relationship is found:

\[
\delta \sigma_2 = v_2 (1 - \gamma / ku) \delta m_2 + \rho_2 g \xi - \gamma \xi \rho_2 v_2 (1 + \gamma / ku).
\]

On the other hand, using the isothermal approximation expressed by Eq. (13), we can write the energy equation behind the front \((y < 0)\) as follows:

\[
\nabla^2 (\delta \rho_1) = 0,
\]

and the mass and momentum equations are

\[
\nabla \cdot (\delta \mathbf{v}_1) = 0, \quad \rho_1 \left( \frac{\partial}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial}{\partial y} \right) \delta \mathbf{v}_1 \approx - \nabla \delta \rho_1.
\]

The previous equations allow us to introduce the streamfunction \( \Psi \) [\( \delta \mathbf{v}_1 = (\partial \Psi / \partial y) \mathbf{e}_y - (\partial \Psi / \partial x) \mathbf{e}_x \)] and the vorticity \( \omega = \nabla \times \delta \mathbf{v}_1 \). Then, we have

\[
\nabla^2 \Psi = - \omega, \quad \frac{\partial \omega}{\partial t} + v_1 \frac{\partial \omega}{\partial y} = 0.
\]

Since all the perturbations are of the form \( \exp(ikx + \gamma t) \), we obtain

\[
\frac{d^2 \Psi}{dy^2} - k^2 \Psi = - \omega_0 e^{-\gamma y} \psi_1,
\]

where \( \omega_0 \) is the vorticity behind the interface and, hereafter, the symbols denote the part of the functions depending on the \( y \) coordinate. Then, the solution of Eq. (54) that vanishes for \( y \to +\infty \) is

\[
\Psi = \Psi_0 e^{-\gamma y} + \frac{\omega_0}{k} e^{-\gamma y} \psi_1.
\]

In order to determine the streamfunction and the vorticity we need to specify the flow velocity at the interface resulting from the flux energy and the momentum conservation equations in the transverse direction \((x \text{ axis})\) and then, it turns out that

\[
-ik \left( \Psi_0 + \frac{\omega_0}{k} \right) = \delta v_{1y} \approx \gamma \xi + \frac{d}{dy} \left( \frac{2 \kappa_{D1}}{5 \rho_1} \delta \epsilon_1 \right) + \frac{\delta m_2}{\rho_2} = \gamma \xi + k \xi \psi_1 [1 + (F - 1) \tau_D],
\]

\[
-i \left( k \Psi_0 + \frac{\gamma \omega_0}{v_1} \right) = i \delta v_{1y} \approx - \gamma \xi + k \xi \psi_1 [1 - (F + 1) \tau_D],
\]

where \( F = \delta m/z (k \xi \rho_2 v_2) \). The mass conservation through the interface yields \( \delta m_2 = \delta m_1 \), and, using Eq. (56), it is straightforward to obtain \( F \approx 1 \). Thus, we obtain the following equations:
\[
\frac{i \omega_0}{k^2 v_1} \left(1 - \frac{\gamma}{kv_1}\right) = 2 \left(\frac{\gamma}{kv_1} + r_D\right).
\]

(58)

\[
\Psi_0 = -\frac{\omega_0}{k^2} \frac{i \zeta}{k} (\gamma + kv_1).
\]

(59)

In addition, the perturbed pressure behind the interface results to be

\[
\delta \rho_1 \approx -\frac{i \rho_1}{k} \left(\frac{\gamma}{kv_1} \frac{\partial \Psi}{\partial y} + v_1 \frac{\partial^2 \Psi}{\partial y^2}\right)
\]

\[
= i \rho_1 \Psi_0 (\gamma - kv_1)
\]

\[
= k \zeta \rho_1 v_1 \left[-1 + \left(\frac{\gamma}{kv_1}\right)^2 + i \frac{\omega_0}{k^2 \zeta v_1} \left(1 - \frac{\gamma}{kv_1}\right)\right].
\]

(60)

and the momentum flux in the hot region \((\gamma > 0)\) relative to the moving ablation surface is

\[
\frac{\delta \rho_1}{k \zeta \rho_1 v_1} \approx 1 + \left(\frac{\gamma}{kv_1}\right)^2 + i \frac{\omega_0}{k^2 \zeta v_1} \left(1 - \frac{\gamma}{kv_1}\right) + \frac{g}{kv_1}.
\]

(61)

Then, from the previous equation and Eq. (50), and considering \(\delta \rho_1 = \delta \rho_2\) and \(\delta \rho_2 = \delta \rho_1\), we can reobtain the dispersion relation given by Eq. (37).

Now, we analyze the physical mechanisms leading to the growth rate reduction predicted by Eq. (37). For this, we multiply Eq. (37) by \(\zeta\) and operate the following substitutions: \(\gamma \zeta \rightarrow d \zeta / dt\) and \(\gamma^2 \zeta \rightarrow d^2 \zeta / dt^2\). Thus, we obtain

\[
\frac{d^2 \zeta}{dt^2} + f_d \frac{d \zeta}{dt} + \Omega^2 \zeta = 0.
\]

(62)

As can be seen, this equation represents an oscillator with a damping coefficient \(f_d\) and a natural frequency \(\Omega\):

\[
f_d = \frac{4 k v_2}{1 + r_D}, \quad \Omega^2 = kg \left(\frac{k v_2^2}{g r_D} - A_T\right).
\]

(63)

The dominant stabilization effect that can lead to complete stability for sufficiently large perturbation wave numbers comes from the rocket effect equilibrating the gravity force. In such a case, \(\Omega^2 > 0\) or, equivalently, \(k \rho_1 v_1^2 > \rho_2 g\). Thus, the following description results from the present model: when a fluid element on the front surface moves into the hot region as a consequence of the perturbation, its temperature is not modified (the front is an isotherm) but the temperature gradient, as well as the energy flux behind the front, increases. This fact produces an increase in the fluid velocity behind the front and the consequent increase of the dynamic pressure. This reaction tends to reduce the perturbation growth and, for relatively short perturbation wave numbers, causes the total stabilization.

On the other hand, the damping force reduces the growth rate and produces the so-called convective stabilization. From Eqs. (58) and (61), and using the mass and momentum conservation equations, we can see the different ingredients that contribute to the damping force:

\[
f_d \approx (1 + 2) k v_2 \approx \left(\frac{\delta m_2}{k \zeta} + \rho_2 v_2 + k \omega_0 \frac{1}{k} \rho_2 v_2\right) \frac{k}{\rho_2}.
\]

(64)

where, for simplicity, we have assumed \(r_D \ll 1\). So, half of the damping effect is produced by the momentum flux and accounts for the vorticity convection, and the other half is produced by the mass flow rate and corresponds to the fire polishing effect: \(\rho_2 v_2 + \delta m_2 / k \zeta\). Let us remark that the picture coming out from the present model is substantially different from that resulting from the Takabe fitting formula,\(^{12,13}\) which suggests that complete stabilization for the short wavelength is caused by convection. We find that, although convective mechanisms contribute to the growth rate reduction, they can never lead to total stabilization, which is instead controlled by the dynamical pressure. Of course, dynamical pressure is interrelated with convection flow and with other phenomena involved in the ablation process, but their roles in the stability of the front are physically different.

For \(\Gamma > 1\) the lateral transport becomes as important as the dynamical pressure in determining the cutoff wave number, and it also gives a contribution to the damping force of the same order as the vorticity convection and fire polishing effects.

V. CONCLUDING REMARKS

We have developed a new model for the Rayleigh–Taylor instability of a steady ablation front based on the SBM. Differently from previous versions of the model, we have considered the thermal flux behind and across the front. This allows us to obtain a self-consistent expression for the growth rate in terms of an unknown parameter, namely the density jump \(r_D\) associated with the instability process. Such a parameter depends on the structure of the flow behind the front and relates to the well-known problem of the missing information when a weak expansion is treated as a discontinuity. The explicit dependence on the thermal conductivity is eliminated by exploiting the fact that, in the discontinuity model, the ablation surface is an isotherm.

By introducing a simple corona model based on a diffusive process of energy transport, and assuming that the instability involves the region within a distance \(y^* = 1/2k\) from the ablation front, the parameter \(r_D\) is calculated. The resulting growth rate is in excellent agreement with more sophisticated self-consistent models and with numerical results in the regime characterized by \(\Gamma \ll 1\). An extension of the model in order to describe the complementary regime \(\Gamma > 1\) is obtained by including the effects of the lateral transport as in Ref. 10, and it turns out to be in very good agreement with numerical calculations.

The numerical calculations and the self-consistent theories above mentioned use a well-defined corona model for the unperturbed situation and we have calculated \(r_D\) by means of the same model in order to make the previous comparisons consistent.

Discrepancies with the fitting formula given by Eq. (46) are found for \(\beta \neq 1.5\), and it can probably be explained by the presence of a suprathermal component in the energy transport process. Therefore, an adequate comparison between that formula and the present model may require the availability of more realistic corona models than the one used here.
and, in any case, we would need to know more details about
the physics underlying the simulations from which the for-

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tude to the Staff of the Department of Applied Physics for
algebra we obtain the following equations for the perturba-
eigenvalue of the system. If this fifth-order system has
l cutoff wave number arises from the increase (decrease)
of the perturbed dynamical pressure on the fluid elements
placed at the interface that move into the hot region (cold
region) as a consequence of the perturbation.

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APPENDIX: STABILITY ANALYSIS OF ANABLATION FRONT WITH SMOOTH GRADIENTS

Here we study the Rayleigh–Taylor instability of a
steady ablation front with smooth gradients by means of
the WKB approximation. We consider that the density gradient
has a characteristic length $L_0$ such as it was defined in Eq.
(39) and we assume small perturbations of wave number $k$
for which the condition $kL_0 \gg 1$ holds. Then, we proceed
with the stability analysis by linearizing the fluid equations
and assuming that every small perturbation $\delta \varphi$ of a quantity
$\varphi(t,x,y)$ is of the form $\delta \varphi = \exp(yt + ikx)$. Thus, after some
algebra we obtain the following equations for the per-

\begin{align}
\gamma \delta \rho + \frac{d}{dy} \left( \delta \rho v_0 + \rho \delta v_y \right) &= -ik \rho_0 \delta v_y, \quad (A1) \\
\rho_0 \left( \gamma \delta v_y + v_0 \frac{d(\delta v_y)}{dy} \right) &= -ik \delta \rho, \quad (A2) \\
\rho_0 \left( \gamma \delta v_y + \frac{d(v_0 \delta v_y)}{dy} \right) + \delta \rho v_0 \frac{dv_0}{dy} &= -\frac{d(\delta \rho)}{dy} + \rho_0 g, \quad (A3) \\
-ik \delta v_y + k^2 \frac{\kappa_D}{\rho_0} \delta \rho &= \frac{d}{dy} \left( \delta v_y + \frac{d}{dy} \left( \frac{\kappa_D}{\rho_0} \delta \rho \right) \right). \quad (A4)
\end{align}

These equations can be combined to obtain a fifth-order dif-
ferential equation for the velocity perturbation $\delta v_y$, with co-
efficients that depend on the coordinate $y$. These coefficients
involve constants such as the growth rate $\gamma$, which is the
eigenvalue of the system. If this fifth-order system has $l_1$
 modes vanishing at $y \to +\infty$ and $l_2$ modes vanishing at $y \to
\infty$, so that $l_1 + l_2 = 5$, then the dispersion relation $\gamma(k)$ is
obtained in the usual way from the conditions for the exis-
tence of nontrivial solutions. In the WKB approximation
the perturbation $\delta v_y$ is written in the form $\delta v_y \approx \exp(\int q \, dz)$, where $q$ are the roots of the following fifth-
order characteristic function:

\begin{align}
P(u, \gamma, k, y) &= \left( \frac{\gamma}{kv_0} + u \right)^2 (1 - u^2) + \frac{k \kappa_D}{\rho_0 v_0} \left( \frac{\gamma}{kv_0} + u \right) \\
&\quad \times (1 - u^2)^2 - G = 0, \quad (A5) \\
G &= -\frac{g}{k^2 v_0} \frac{d\ln(\rho_0)}{dy}, \quad u = \frac{q}{k}. \quad (A6)
\end{align}

where we have considered that the coefficients of the fifth-
order differential equation depend weakly on $y$.

On the other hand, in order to obtain nontrivial solutions,
we have to require the existence of two turning points, at
$y = y_{1T}$ and $y = y_{2T}$, where the WKB solution breaks down:

\begin{align}
\frac{\partial P}{\partial u} &= 0, \quad (A7) \\
\frac{\partial P}{\partial y} &= 0. \quad (A9)
\end{align}

Thus, Eqs. (A5)–(A9) represent the formal solution of the
problem. For simplicity, and taking into account that
$k \kappa_D / \rho_0 v_0 \gg 1$, we retain the first term of $P(u, \gamma, k, y)$ only
in Eq. (A5) and we neglect it in Eqs. (A7) and (A9). In this
way, the following solution is found:

\begin{align}
\frac{\gamma}{kv_0} &= \frac{1 - 5u^2}{4u}, \quad (A10) \\
G &= \left(1 - u^2\right)^3 \left(1 + 4u \frac{k \kappa_D}{\rho_0 v_0}\right), \quad (A11) \\
u^2 &= \frac{1 - F(\theta)}{5 - F(\theta)}, \quad F(\theta) = \frac{(2
\nu + 2) \theta - (2 \nu + 3)}{\theta - 1}, \quad (A12)
\end{align}

where $\theta = \epsilon_1 / \epsilon_2$ (with $\epsilon_i$ representing the specific internal
energy ahead of the ablation front), $G$ is given by Eq. (A6),
and we have used the corona model mentioned in Sec. III for
the zeroth-order solution.\cite{14,21}

\begin{align}
P_0 &= \frac{2}{3} \rho_0 v_0 \approx p_2; \quad \bar{m} = \rho_0 v_0; \quad (A13) \\
\chi \bar{e}_0 \approx \frac{5}{3} \bar{m} (\bar{e}_0 - \bar{e}_2)
\end{align}

where $p_2$ is the ablation pressure.
From the previous equations we can find the dispersion relation \( \gamma(k) \) with \( \Gamma = 3g \kappa D_s / 5 \rho \nu^{3/2} \) as a parameter, and it turns out in good agreement with the numerical calculations reported in Ref. 15. Here, we want to emphasize the essential role of the thermal conduction and, in particular, the significant contribution of the lateral thermal transport represented by the second term on the left side of Eq. (A4). In fact, from this equation we see that the relative importance of the thermal conduction across the front with respect to the lateral transport is given by the value of \( \nu^2 = q^2 / k^2 \). For perturbation wave numbers close to the cutoff value \( k_c \), we have, from Eq. (A10), that \( \nu^2 \approx 1 / 2 \), indicating that lateral transport dominates the thermal conduction. We can also find the following analytical expression for the cutoff wave number from Eqs. (A6), (A9), and (A11):

\[
\kappa^3 \Gamma^2 + \frac{5}{4} \left( \frac{2 \nu + 2}{2 \nu + 3} \right)^\nu \kappa^2 \Gamma - \frac{7 (2 \nu + 2)^2 \nu \nu^2}{2 (2 \nu + 3)^2} = 0, \quad k_c \nu^2 = \frac{g}{h}.
\]  
(A14)

For relatively large values of \( \Gamma \), the previous equation yields

\[
k_c = h(\nu) \Gamma^{-2/3}, \quad h(\nu) \approx 1.5 \left( \frac{2 \nu + 2}{2 \nu + 3} \right)^{(2 \nu + 2)/3}.
\]  
(A15)

This expression shows two interesting facts. First, the asymptotic behavior of \( k_c \) is a power law with an exponent independent of \( \nu \) and, second, \( k_c \) is a decreasing function, although weakly, of \( \nu \) denoting a tendency opposite to that observed for \( \Gamma < 1 \).