Linear perturbation growth at the trailing edge of a rarefaction wave

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An analytic model for the perturbation growth inside a rarefaction wave is presented. The objective of the work is to calculate the growth of the perturbations at the trailing edge of a simple expanding wave in planar geometry. Previous numerical and analytical works have shown that the ripples at the rarefaction tail exhibit linear growth asymptotically in time [Yang et al., Phys. Fluids 6, 1856 (1994), A. Velikovich and L. Phillips, ibid. 8, 1107 (1996)]. However, closed expressions for the asymptotic value of the perturbed velocity of the trailing edge have not been reported before, except for very weak rarefactions. Explicit analytic solutions for the perturbations growing at the rarefaction trailing edge as a function of time and also for the asymptotic perturbed velocity are given, for fluids with $\gamma<3$. The limits of weak and strong rarefactions are considered and the corresponding scaling laws are given. A semi-qualitative discussion of the late time linear growth at the trailing edge ripple is presented and it is seen that the lateral mass flow induced by the sound wave fluctuations is solely responsible for that behavior. Only the rarefactions generated after the interaction of a shock wave with a contact discontinuity are considered. © 2003 American Institute of Physics. [DOI: 10.1063/1.1618773]

I. INTRODUCTION

The dynamical evolution of simple expanding waves or rarefactions is certainly a basic issue in fluid dynamics and is important in several fields, among them inertial confinement fusion (ICF).1,2 In a typical laser fusion experiment, the irradiated material undergoes an initial compression by a strong shock which is typically followed by an expansion wave. The rarefaction wave is formed either when the shock arrives to the rear surface (which generally limits with vacuum) or when it crosses any other contact surface at which the density jump is negative. This is a common scheme adopted in ICF targets driven by laser in order to shape the entropy profile for successful ignition of the fuel at the time of spherical stagnation.2 This initial shaping will lead, later in time, to the formation of a hot central volume in which thermonuclear reactions will burn a fraction of the fuel providing high energy gain. Therefore, the process of entropy shaping needs the generation of the shock-rarefaction sequence, either in direct or indirect drive irradiation. The presence of ripples at any of the surfaces at which the shocks or the rarefactions are generated will induce the generation of pressure and velocity perturbations in the bulk of the target, degrading the overall performance of the implosion. These perturbations are the seeds for hydrodynamic instabilities which would damage the spherical symmetry of the target, reducing its energy gain. Among the most studied instabilities in this context, we mention the shock induced Richtmyer–Meshkov instability (RMI).3–6 and the gravity driven Rayleigh–Taylor instability (RTI) at the ablation front of laser fusion targets.5,7 Both instabilities have received considerable attention and numerous theoretical and experimental studies are still being carried on, in order to improve our understanding of perturbation evolution inside the irradiated targets. Regarding the RMI, the reader is referred to the earliest and also to the most recent experimental works done with high power lasers and which were aimed to get a deeper understanding of the interaction between the contact discontinuity and the deformed shock and rarefaction fronts.8–10 Also, the reader is referred to comprehensive reviews of the state of the art of the hydro-instability problem as reported at major laboratories.11–14

In this work we will consider the basic problem of the evolution of the perturbations that originate inside a planar rarefaction wave. Despite this problem having been attacked several times in the recent past, no complete analytical scaling laws for the perturbations rate of growth at the trailing edge were available. As it is well known, fluid quantities at both ends of the rarefaction fan are continuous but not their normal derivatives.15–18 Simple expanding waves or rarefactions are also called, because of this, weak discontinuities.15 Due to the continuity of the density both at the leading and at the trailing fronts, the ripples imposed on those surfaces are quite hard, if not impossible, to observe experimentally. They are also quite difficult to follow in the numerical simulations.17 Due to these facts, the theoretical understanding provided by more accurate and exact analytical models becomes a valuable tool with which to improve our knowledge of perturbation evolution, providing at the same time a good feedback mechanism between experimental and theoretical work. The perturbations usually dealt with in ICF applications are sufficiently small so that a linear theory description is reasonable. Therefore, one is tempted to conclude that the results of such simplified descriptions would be of easy interpretation. However, as has been learned in the past decades, the reverse has proven to be true. Even more so, when compressibility effects are important to describe instability evolution, as usually happens in RMI environments. In this work we will concentrate on the perturbation evolution inside a rarefaction wave, where sound pressure perturba-
tions and the subsequent perturbed lateral mass flow induced by them are the key ingredients necessary to understand perturbation growth. These results could certainly be helpful as a complementary tool to two-dimensional hydrodynamic simulation codes such as those used in typical studies of the RMI for laser fusion targets.5–14 In addition, and perhaps as important as the previously mentioned reason, the development of more elaborate analytical models is expected to provide a deeper understanding of the basic mechanisms that drive perturbation growth. This is particularly true in the case of the rippled rarefaction waves, where at large enough times, and in spite of the fact that the sound waves perturbation field has decayed considerably inside the expanding fluid, the ripples imposed on its trailing front continue growing in a way that resembles very much the RM growth at a fluid, the ripples. The approximate growth rate of the trailing edge ripples.

We restrict our discussion to rarefactions generated as the result of the interaction between an incident shock and a contact surface separating two different materials. Let us assume that an incident planar shock comes from the right (see Fig. 1) toward the interface separating fluids “a” and “b.” The pre-shock density of fluid “b” is $\rho_{b0}$ and its compressed value is $\rho_{b1}$. The fluid velocity in the laboratory reference frame, behind the incident shock is $-v_1 \hat{x}$. The initial density of the fluid to the left of the contact surface is $\rho_{a0}$. We assume that the shock front arrives to the contact surface at $t=0$ and that at $t=0^+$ another shock is launched into fluid “a.” The contact discontinuity serves as a piston that launches a shock in the left fluid.5,15 If the thermodynamic properties of both fluids are properly chosen, a rarefaction is reflected inside fluid “b.” If $\gamma_a = \gamma_b$, then $\rho_{a0} < \rho_{b0}$ is enough to ensure a rarefaction reflected back into fluid “b.” If the isentropic exponents are not equal, the conditions on the density jump to ensure a reflected rarefaction become slightly more complicated and it depends on whether $\gamma_a < \gamma_b$ or not. For details about the necessary conditions in order to have a reflected rarefaction, the interested reader can always refer to the works of Velikovich and Phillips or Yang et al.17,18 At $t=0^+$ we assume that the transmitted shock and the reflected rarefaction are fully formed. According to Fig. 1, the transmitted front moves to the left with velocity $-u_i \hat{x}$ and the contact discontinuity moves with velocity $-v_1 \hat{x}$. The densities at both sides of it are $\rho_{af}$ to the left and $\rho_{bf}$ to its right. The centered expansion wave moves to the right and is bounded by two fronts moving at different velocities. The rarefaction head or leading edge, moves to the right with the local sound speed, which in the laboratory frame of reference amounts to $c_{bf} - v_1$. The trailing edge, or rarefaction tail, moves also to the right, but with the speed $c_{bf} - v_1$. Density, pressure, temperature, and all the thermodynamic quantities of fluid “b” change continuously from the leading to the trailing edge. The rarefaction strength is characterized as in Velikovich and Phillips,17 with the ratio of the downstream and upstream sound speeds: $M_1 = c_{bf}/c_{b1}$.

We assume that the material interface has an initial corrugation with amplitude $\psi_0$ and wavenumber $k = 2\pi/\lambda$, where $\lambda$ is the perturbation wavelength. A linear theory approximation is assumed: $\psi_0 \ll \lambda$. The scenario depicted in Fig. 1 occurs naturally in many environments in which a shock is launched toward a contact surface, as is usual in the first stages of the irradiation of laser fusion targets. The initial corrugation at the interface is propagated into the bulk of both fluids either by the transmitted shock front or by the corrugated rarefaction wave. For $t>0^+$ the fronts interact with the contact surface through sound waves. This sonic interaction makes the interface ripples oscillate at first and later on achieve a constant velocity, which is known as the RMI.5–5 For the picture drawn in Fig. 1, the perturbations at the material surface are fed with perturbations originating inside the rarefaction fan and at the shock front. The sound waves originating at the shock front travel up to the interface where they are refracted. Those escaping back to the shock will be reflected at the shock from behind and will arrive again to the interface at a later time. However, those perturbations transmitted into fluid “b” will never reach the rarefaction tail, because the trailing front of the rarefaction moves with the same speed as the pressure fluctuations. On the other side, the perturbations that originate inside the rarefaction, travel downwards in the negative $x$ direction, arrive to the rarefaction tail, and get directed toward the contact surface. The situation, roughly speaking, is that the shock and contact surface perturbations can have no effect on the rarefaction evolution for $t>0^+$. However, the rarefaction perturbations have a definite causal influence on the evolution of the perturbations growing at the material interface and the shock. Besides, due to the adiabaticity of the motion...
inside the expansion wave, neither vorticity nor additional entropy are generated inside the expanding fluid. That is, within the limits of ideal hydrodynamics, the entropy and vorticity modes of perturbation are not excited inside the rarefaction fan. Only sound modes are of interest to the perturbation problem, and among them, only those that propagate downwards, from the rarefaction head to the rarefaction tail. These facts will be seen to greatly help with the algebra of the perturbation equations inside the expanding fluid. Once the perturbed rarefaction starts to move to the right, the perturbations that originate at the leading front propagate downstream and reach the trailing edge. It has been shown in the last decade that the corrugations at the trailing edge also exhibit a temporal growth similar to the growth that occurs at the contact surface,17,18 in much the same way as the RMI. This behavior was explicitly shown in the work of Yang et al.,18 and was studied analytically by Velikovich and Phillips.17 They studied the perturbations of a rarefaction generated by a strong shock or by an initial pressure discontinuity as a function of the rarefaction strength. A series solution for the perturbation velocity at the trailing edge was found, but no closed formulas in the general case were obtained, except in the limit of a weak rarefaction.

In this work, we develop an analytical framework with which to study the perturbations evolution inside the rarefaction, focusing our attention on the growth of the rarefaction tail ripples. Our solution is based on previous calculations of Kivity and Hanin,20,21 and we will get explicit analytical laws either for the growth rate as a function of time, or for its asymptotic value for any rarefaction strength. Thus, our results complement and extend those of the previously mentioned works.17,18,20 The work is organized as follows: In Sec. II the zero-order profiles are derived. In Sec. III the equations of motion are linearized and a change of variables is suggested to simplify the discussion. A telegraph equation is deduced for the tangential velocity perturbations. The equation is solved and the general form of the perturbed quantities is given as a function of time and position inside the rarefaction. In Sec. IV the equation for the perturbation at the trailing front is solved. The asymptotic form of the velocity is deduced as a function of the fluids parameters. In Sec. V the results are compared with those existing in the literature and the asymptotic growth rate is studied as a function of the rarefaction strength for different combinations of the initial fluids parameters. A semi-qualitative discussion is intended in Sec. VI, in an attempt to understand the late time linear growth observed at the rarefaction tail ripples. The question is obviously not closed with the interpretation provided there, and it is hoped that it could give some useful insights toward a better understanding of the subject. Finally a short summary is presented in Sec. VII.

II. UNPERTURBED PROFILES

In this section we calculate briefly the zero-order profiles of the rarefaction as a function of the initial fluid parameters. Four parameters are necessary to characterize the interaction between the incident shock front and the contact surface together with the subsequent generation of the rarefaction and the transmitted front. Namely, the initial shock front intensity, defined here as $z_i = (p_1 - p_0)/p_0$, the initial density ratio at the material interface: $R_0 = p_0/\rho_{b0}$, and finally, the isentropic exponents of both fluids: $\gamma_a$ and $\gamma_b$.

The shock is coming from fluid “b” toward the left. The unshocked fluid has density $\rho_{b0}$ and the compressed fluid has density $\rho_{b1}$. The different magnitudes of interest across the incident shock can be obtained from the conservation equations.15,16 The density compression across the shock front can be written as

$$\frac{\rho_{b1}}{\rho_{b0}} = 1 + \frac{\epsilon_{b1} z_i}{\gamma_b \sqrt{1 + \epsilon_{b1} z_i}},$$

where $\epsilon_{b1,2} = (\gamma_b \pm 1)/(2 \gamma_b)$. The incident shock Mach number is given by $M^2 = 1 + (\gamma_a + 1)/(2 \gamma_a) z_i$. Besides, the fluid velocity behind the incident shock front, in the laboratory reference system, is given by

$$v_i = \frac{z_i}{\gamma_b \sqrt{1 + \epsilon_{b1} z_i}} c_{b0}.$$

The incident shock velocity is

$$u_i = c_{b0} \sqrt{1 + \epsilon_{b1} z_i},$$

and the sound speed of the compressed fluid is

$$c_{b1} = c_{b0} \sqrt{1 + \frac{z_i}{\mu_{b1}}}.$$

The shock arrives to the interface at $t = 0$ and we assume that in a very small time interval, the transmitted shock front and the reflected rarefaction are formed at each side of the contact surface. As we did with the incident shock, the transmitted front intensity is $z_i = (p_2 - p_0)/p_0$, where $p_2$ is the pressure driving the transmitted shock wave, which is assumed to be the same at both sides of the contact surface. We must calculate the zero-order quantities inside the rarefaction to develop later on the perturbation model. We need to couple both fluids across the interface requiring continuity of pressure and normal velocity. At the transmitted front, the equations are similar to the incident shock calculations. On the side of the expanding fluid, we can use the constancy and uniformity of one of the Riemann invariants (see, for example, the book of Zeldovich and Raizer16) (pages 15–38). After requiring the continuity of pressure and normal velocity at the interface separating both materials, we get

$$\frac{z_i}{\gamma_a \sqrt{1 + \epsilon_{a1} z_i}} = \sqrt{\frac{\gamma_a R_0}{\gamma_b}} \frac{z_i}{\gamma_b \sqrt{1 + \epsilon_{b1} z_i}} + \frac{2(1 - M_1)^2}{\gamma_b - 1} \sqrt{1 + \frac{z_i}{\mu_{b1}}},$$

where we used

$$M_1 = \left(\frac{1 + z_i}{1 + \frac{z_i}{\gamma_b (\gamma_b - 1)}}\right)^{(\gamma_b - 1)/(2 \gamma_b)}.$$

Equations (5) and (6) must be solved together and then we obtain $M_1$ or $z_i$. Once we have any one of them, we can
calculate the remaining quantities pertaining to the fluid “$a$.” In particular, the density compression ratio across the transmitted front is

$$
\mu_{a2} = \frac{\rho_{a2}}{\rho_{a0}} = \frac{1 + e_{a1} z_t}{1 + e_{a2} z_t},
$$

(7)

where $e_{a1,2} = (\gamma \pm 1)/(2 \gamma_a)$. The transmitted shock front speed in the laboratory reference frame is

$$
u_t = \sqrt{1 + e_{a1} z_t} c_{a0},
$$

(8)

and the contact surface velocity is

$$
u = \frac{z_t}{\gamma_a \sqrt{1 + e_{a1} z_t}} c_{a0}.
$$

(9)

The final sound speed in fluid “$a$” is given by

$$c_{af} = c_{a0} \sqrt{1 + z_t / \mu_{a2}}.
$$

(10)

We see from Eq. (5) that for given $R_0$, $\gamma_a$, and $\gamma_b$, $M_1$ is a function of the incident shock intensity. It will be shown that $M_1$ is a decreasing function of $z_t$ when the other parameters remain fixed. For a shock of infinite strength we would arrive to the minimum possible value of $M_1$, which we call $M_{1 \text{min}}$. We cannot give the exact analytical solution for $M_{1 \text{min}}$ for an arbitrary choice of $R_0$, $\gamma_a$, and $\gamma_b$. We can show instead an iterative procedure with which to get its value with sufficient accuracy.

That is, the minimum possible value of $M_1$ for a given set of values for $R_0$, $\gamma_a$, and $\gamma_b$ is the result of

$$M_{1 \text{min}} = \lim_{z_t \to \infty} M_1(R_0, \gamma_a, \gamma_b, z_t).
$$

(11)

Therefore, we take the limit $z_t \to \infty$ in Eq. (5) and construct the following iteration sequence:

$$M_{1\text{[0]}} = 1,
$$

(12)

$$M_{1\text{[k+1]}} = (\gamma_a \gamma_b R_0) \left( \frac{1}{\gamma_b} \right) \frac{e_{a1}}{e_{b1}} \left( \frac{1}{2} + \frac{2 \sqrt{e_{a1} e_{b1}} \sqrt{\gamma_b^{-1} \left(1 - M_{1\text{[k]}}^{-1}\right)}}{\gamma_b^{-1} \left(1 - M_{1\text{[k]}}^{-1}\right)} \right)^{\gamma_b^{-1}/\gamma_b},
$$

where $k$ is the index of the iteration chain.

The zero-order pressure ($p_2$), density ($\rho_{a2}$), fluid velocity ($\nu_t$), and all the other thermodynamical variables are uniform and constant in the space between the transmitted shock front and the contact surface. The same happens between the contact discontinuity and the rarefaction trailing edge ($p_{bf}$, $\rho_{bf}$, and $\nu_r$, respectively). Inside the rarefaction, the magnitudes change continuously from their leading front values to their trailing front values.

The sound speed at the trailing edge of the rarefaction is therefore $c_{bf} = c_{b1} M_1$. The thermodynamical variables inside the expanding fluid are functions of time and space through the self-similar combination $x/t$. We define the self-similar variable $\xi = x/(c_{b1} t)$. It can be seen that the following relation holds:

$$c_{b1} \xi = \nu + c,
$$

(13)

where $\nu$ is the fluid velocity inside the rarefaction and $c$ is the sound speed. It is found convenient to work with the variable $A = c/c_{b1}$ as defined in Velikovich and Phillips.17 It characterizes each slide inside the rarefaction by its relative sound speed with respect to the sound velocity at the front. There is a linear relationship between $A$ and $\xi$:

$$A = \frac{c}{c_{b1}} = \frac{2}{\gamma_b + 1} \xi.
$$

(14)

The density and pressure, normalized with respect to their values at the rarefaction leading front, are given by

$$\tilde{\rho} = \frac{\rho}{\rho_{b1}} = A^{2/(\gamma_b - 1)},
$$

(15)

$$\tilde{p} = \frac{p}{p_{b1}} = A^{(2 \gamma_b)/(\gamma_b - 1)}.
$$

(16)

We also have some algebraic relationships that will prove to be useful later on:

$$\frac{d v_s}{d \xi} = \frac{2 c_{b1}}{\gamma_b - 1},
$$

(17)

$$\frac{d A}{d \xi} = - \frac{\gamma_b - 1}{\gamma_b + 1},
$$

(18)

$$\frac{d \rho}{d \xi} = \frac{2}{\gamma_b + 1} \rho.
$$

(19)

### III. PERTURBED PROFILES

#### A. Linearized equations inside the rarefaction

To work with the perturbations originating inside the rarefaction, it is convenient to follow the model of Kivity and Hanin.20,21 We begin by writing the fluid equations for an inviscid fluid, that is, the conservation of mass, linear momentum, and energy:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,
$$

(20)

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = - \frac{\nabla p}{\rho},
$$

(21)

$$\frac{\partial s}{\partial t} + (\mathbf{v} \cdot \nabla) s = 0,
$$

(22)

where $\mathbf{v}$ is the fluid velocity and $s$ is the fluid entropy. The last equation (22) states that the entropy of the fluid particles remains constant as the particle moves inside the expansion wave. This means that for any fluid particle inside the rarefaction, the relationship $p \propto \rho s$ does hold. If the fluid in front of the expansion has no perturbations in entropy, then the particles will conserve their initial values of the entropy up to the time when they exit the rarefaction region from behind. The same property can also be seen for the vorticity.20,21 That is, the rarefaction region is an entropy and vorticity preserving region for the fluid. These two constants of motion will be of great help later, in order to simplify the
mathematical treatment of the perturbations inside the expanding fluid. We work in a system of reference fixed to the contact surface. Then, the rarefaction head moves with dimensionless speed \( \xi_{th} = 1 + (v_1 - v_2)/c_{b1} \) and the rarefaction front moves with dimensionless velocity \( \xi_n = M_1 = c_{bf}/c_{b1} \).

We linearize the previous system of equations taking into account the following definitions: the total velocity in the \( x \) direction is written as \( v_x + \delta v_x \), where \( v_x \) refers to the unperturbed profile calculated in the previous section and \( \delta v_x \) to its perturbation. On the contrary, for the tangential direction we only have the perturbed velocity \( \delta v_y \). We further make

\[
\delta v_x = c_{b1} \delta u(x,t) k \psi_0 \cos ky, \tag{23}
\]

\[
\delta v_y = c_{b1} \delta v(x,t) k \psi_0 \sin ky, \tag{24}
\]

and it is not difficult to verify that the perturbations in pressure (\( \delta p \)) and density (\( \delta \rho \)) both depend on the lateral coordinate as \( k y \). In fact, we define

\[
\delta \rho = \rho_{b1} \delta \rho / k \psi_0 \cos ky. \tag{25}
\]

All the perturbed quantities are linear in \( k \psi_0 \). For a homogeneous fluid ahead of the rarefaction front, there are no entropy perturbations and then, the perturbed energy equation decouples from the other equations. We also use the fact that the vorticity is conserved and that its value is equal to zero, assuming that the head of the rarefaction is irrotational. We work with the dimensionless time \( \tau = kc_{b1}t \). The linearized equations of motion can therefore be written as

\[
\frac{\partial \dot{\rho}}{\partial \tau} - \frac{A \partial \dot{\rho}}{\partial \xi} + \frac{\dot{\rho} \partial \dot{\rho}}{\partial \xi} + \frac{2}{\gamma_0 + 1} \frac{1}{\tau} \delta \rho + \dot{\rho} \delta v = 0, \tag{26}
\]

\[
\delta u = - \frac{1}{\tau} \frac{\partial \dot{\rho}}{\partial \xi}, \tag{27}
\]

\[
\delta \rho = \frac{\dot{\rho}}{A^2} \frac{\partial \dot{\rho}}{\partial \tau} - \frac{\dot{\rho}}{A \tau} \frac{\partial \dot{\rho}}{\partial \xi}. \tag{28}
\]

The last equations express the conservation of mass, the condition of zero vorticity, and the conservation of tangential momentum, respectively.

After some algebra, the former system of equations can be substituted by a single equation for the tangential velocity:

\[
\tau^2 \frac{\partial^2 \delta v}{\partial \tau^2} + 2 \left( \frac{\gamma_0 - 1}{\gamma_0 + 1} \right) \tau \frac{\partial \delta v}{\partial \tau} - 2 \left( \frac{\gamma_0 - 1}{\gamma_0 + 1} \right) \tau \frac{\partial}{\partial \tau} \left( \frac{\partial \delta v}{\partial A} \right) = -\tau^2 A^2 \delta v, \tag{29}
\]

where we have made a variable’s change, from \( \xi \) to the variable \( A \), and have made use of Eq. (18).

The last equation can be further simplified by changing to the following new variables:

\[
\eta = \tau^2 A^\alpha, \tag{30}
\]

\[
\xi = A^\beta, \tag{31}
\]

where the values of \( \alpha \) and \( \beta \) will be chosen so as to simplify as much as possible our Eq. (29). It can be seen that the following relationships hold between the old and the new variables:

\[
\tau \frac{\partial}{\partial \tau} = 2 \eta \frac{\partial}{\partial \eta}, \tag{32}
\]

\[
A \frac{\partial}{\partial A} = \alpha \eta \frac{\partial}{\partial \eta} + \beta \xi \frac{\partial}{\partial \xi}. \tag{33}
\]

We substitute Eqs. (32) and (33) into Eq. (29) to get

\[
4 \left[ 1 - \alpha \left( \frac{\gamma_0 - 1}{\gamma_0 + 1} \right) \right] \eta \frac{\partial}{\partial \eta} \left( \eta \frac{\partial \delta v}{\partial \eta} \right) + 2 \left( \frac{\gamma_0 - 3}{\gamma_0 - 1} \right) \eta \frac{\partial \delta v}{\partial \eta} - 4 \beta \left( \frac{\gamma_0 - 1}{\gamma_0 + 1} \right) \xi \eta \frac{\partial}{\partial \eta} \frac{\partial \delta v}{\partial \xi} = - \eta A^{2-\alpha} \delta v. \tag{34}
\]

We see that the first term in the last equation cancels out if we choose \( \alpha = (\gamma_0 + 1)/(\gamma_0 - 1) \). A further simplification can be done by changing to a different unknown which we call \( \delta w \):

\[
\delta w = \xi^{-\alpha} \delta v. \tag{35}
\]

Substituting the above definition, we see after some additional algebra, that by choosing \( \epsilon_0 = -1/2 \) and \( \beta = (\gamma_0 - 3)/(\gamma_0 - 1) \), Eq. (44) takes the following form:

\[
\frac{\partial}{\partial \eta} \frac{\partial \delta w}{\partial \eta} + \frac{4}{3 - \gamma_0} \frac{\gamma_0 + 1}{\gamma_0 - 1} \delta w = 0. \tag{36}
\]

This is the same telegraph equation as obtained by Kiviaty and Hanin. We have derived it here using Eulerian coordinates. To solve it, we proceed along simpler arguments. As was done in a previous work, we recognize that the variable \( \eta \) is essentially the time. Then we take the Laplace transform of the previous equation in the domain of variation of \( \eta \). Accordingly, we define

\[
\delta W = \int_0^\infty \delta w e^{-\sigma \eta} d \eta, \tag{37}
\]

as the Laplace transform of \( \delta w \). Laplace transforming Eq. (36), we obtain a differential equation of the first order in \( \xi \):

\[
\sigma \frac{\partial \delta W}{\partial \xi} + \frac{4}{3 - \gamma_0} \frac{\gamma_0 + 1}{\gamma_0 - 1} \delta W = \frac{\partial}{\partial \xi} \delta w_0, \tag{38}
\]

where the definition of \( \delta w_0 \) is

\[
\delta w_0 = \frac{\delta v(\xi, t = 0 +)}{\sqrt{\xi}}. \tag{39}
\]

We remember that \( \delta v \) is the tangential velocity measured in units of \( c_{b1} \), the sound speed at the rarefaction leading front. The function \( \delta v \) is not uniform inside the rarefaction at \( t = 0 + \), because, as discussed in Velikovich and Phillips, the rarefaction is generated at different instants of time as the incident shock arrives to the corrugated contact surface. The details of the derivation of the initial tangential velocity profile can be found in that reference. We show here its analytical form, as a function of the variable \( \xi \):
\[ \delta v(\xi, t = 0+) = \left[ \frac{1}{\gamma_b + 1} \frac{\psi_0^0 - \psi_0^b}{\xi_{th} - \xi_n} (\xi_{th} - \xi^2) \right. \\
- \left. \frac{2}{\gamma_b + 1} \left( \frac{\psi_0^0 - \psi_0^b}{\xi_{th} - \xi_n} + \psi_0^b \right)(\xi_{th} - \xi) \right]. \]

(40)

The quantities \(\psi_0^0\) and \(\psi_0^b\) are the initial corrugations of the leading and trailing fronts of the rarefaction wave normalized with respect to \(\psi_0\), the pre-shock amplitude of the contact surface ripple. It is not difficult to see that their values are

\[ \psi_0^0 = \frac{\psi_0}{\psi_0} \left(1 + \frac{c_b v_1}{u_i} \right). \]  

(41)

\[ \psi_0^b = \frac{\psi_0}{\psi_0} \left(1 + \frac{c_b v_1}{u_i} \right). \]  

(42)

The initial amplitude of the trailing edge ripple depends on the thermodynamic properties of both fluids, for any shock intensity. That is, the initial tangential velocity profile inside the rarefaction wave does also depend on the thermodynamical properties of both fluids, not only on those of the expanding fluid.

We can integrate Eq. (38) by the method of the variation of parameters:23

\[ \delta W(\xi, \sigma) = \int_{1}^{\xi} \frac{\delta w_1(z) \sigma}{\sigma} e^{-n(\xi - z)/\sigma} \, dz, \]  

(43)

where \(n = (\gamma_b + 1)/(3 - \gamma_b)\) and \(\delta w_1 = d\delta W_0\). We have also assumed that the fluid ahead of the rarefaction leading front is homogeneous and has no velocity perturbations in the tangential direction. After evaluating the inverse Laplace transform of the last expression, we get

\[ \delta v(\xi, \eta) = \sqrt{\xi} \int_{1}^{\xi} \frac{\delta w_1(z)}{\sqrt{n} \eta(\xi - z)} \delta w_1(z) \, dz. \]  

(44)

which is the same solution as obtained by Kivity and Hanin.20,21

We give now the explicit form of \(\delta w_0\) as a function of its arguments. Using Eqs. (14), (31), and (35) together with the previous equation (40), we can write the function \(\delta w_0\) as follows:

\[ \delta w_0(z) = \sum_{j=0}^{2} a_j z^j, \]  

(45)

where the coefficients \(a_j\) are given by

\[ a_0 = \frac{\xi_{th}^2}{\xi_{th} - \xi_n} \left( \frac{\psi_0^0 - \psi_0^b}{\xi_{th} - \xi_n} \right) \left( \frac{\psi_0^0 - \psi_0^b}{\xi_{th} - \xi_n} \right) M_1 \]  

\[ - 4 \frac{1}{\gamma_b - 1} \frac{\psi_0^b}{\xi_{th} - \xi_n} + \frac{\psi_0^b}{\xi_{th} - \xi_n}, \]  

\[ a_1 = \frac{1}{\gamma_b - 1} \left( \frac{\psi_0^0 - \psi_0^b}{\xi_{th} - \xi_n} \right) M_1 + 2 \frac{\psi_0^b}{\xi_{th} - \xi_n}, \]  

\[ a_2 = - \frac{4 M_1}{\gamma_b - 1} \left( \frac{\psi_0^0 - \psi_0^b}{\xi_{th} - \xi_n} \right) M_1, \]  

(46)

and the exponents with \(j = 1, 2\) are

\[ \epsilon_1 = - \frac{1}{2} \frac{\gamma_b - 1}{\gamma_b - 3}, \]  

\[ \epsilon_2 = \frac{3}{2} \frac{\gamma_b - 1}{\gamma_b - 3}, \]  

and the exponent \(\epsilon_0 = -1/2\), as discussed previously.

FIG. 2. Dimensionless tangential velocity as a function of the variable \(A = c/c_{th}\), inside the rarefaction. Three different times are shown. \(\tau = 0+, 8\), and 13. The gases parameters are: \(\gamma_s = 5/3, \gamma_b = 7/5, R_b = 100/725\). The incident shock Mach number is \(M_s = 29.3\) and the rarefaction strength is \(M_r = 0.8448\).

\[ a_1 = \left( \frac{1}{\gamma_b - 1} \right) \left( \frac{\psi_0^0 - \psi_0^b}{\xi_{th} - \xi_n} \right) M_1 + 2 \frac{\psi_0^b}{\xi_{th} - \xi_n}, \]  

\[ a_2 = - \frac{4 M_1}{\gamma_b - 1} \left( \frac{\psi_0^0 - \psi_0^b}{\xi_{th} - \xi_n} \right) M_1, \]  

(47)

\[ \epsilon_1 = - \frac{1}{2} \frac{\gamma_b - 1}{\gamma_b - 3}, \]  

\[ \epsilon_2 = \frac{3}{2} \frac{\gamma_b - 1}{\gamma_b - 3}, \]  

(48)

In Fig. 2 we show the behavior of the tangential velocity perturbations as a function of the variable \(A\) at different times, after shock refraction at the material interface. In Fig. 3 we plot the tangential velocity perturbations as a function of time for different values of the variable \(A\) inside the rarefaction.
efaction region. The fluids parameters chosen here are: \( \gamma_a = 5/3 \), \( \gamma_b = 7/5 \), and \( R_0 = 100/725 \). Figure 4 shows the behavior of the tangential velocity perturbations as a function of the variable \( A \) for different times, for the same pair of gases but with a much lower preshock density ratio: \( R_0 = 10^{-10} \).

We see that when the density jump is finite at the material interface, the velocity perturbations vanish asymptotically in time, at any position inside the rarefaction fan. The magnitude of the tangential velocity perturbation is the greatest at the rarefaction tail initially. Later on, it shows damped oscillations which decrease in time. It can be shown that, at the trailing edge, the perturbations in tangential velocity behave asymptotically as \( J_1(f(M_1, \gamma_b) \tau) / \tau \) where \( f \) is a function of \( \gamma_b \) and \( M_1 \). The reason for such a decay lies in the fact that the sound waves that arrive there are transmitted to the left, inside fluid \( a \). They cannot be reflected back into the expanding fluid. The shape of the corrugated trailing edge would adjust itself, so as to ensure the conservation laws at both sides. In the exact limit of expansion against vacuum, there is no fluid on the left to transmit any sound waves. Moreover, the pressure fluctuations never reach the trailing front, which means that the initial perturbation there does not change in time. For a very low density to the left of the front, which means that the initial perturbation there does not grow nor decrease in time, if the fluid ahead of the rarefaction wave is homogeneous. The perturbations that exist in the whole rarefaction fan [given, for example, as tangential velocity perturbations since \( t = 0^+ \), see Eq. (39)] never reach the rarefaction head. Therefore, there is no way to change the value of the rarefaction head ripples as there are no sound pressure fluctuations coming either from ahead or behind that front. This is not the situation for the rarefaction tail, which simply acts as a surface that transmits all the perturbations coming from the fluid layers in front of it. The perturbations that are generated inside the expansion wave make their way up to the rarefaction tail and are finally transmitted into fluid “\( b \)” to reach the contact surface at a later time. The rarefaction tail ripples evolve in order to ensure the continuity of total density and velocity at both sides of that front. Making use of the conservation laws at the position of the perturbed rarefaction tail it is possible to obtain the evolution equation for their amplitude. We will not repeat the details of the calculations here as they can be found elsewhere. The evolution equation for the trailing edge corrugations can be written as

\[
\tau \frac{d \tilde{u}_n}{d \tau} = \tilde{u}_n + \tau \delta u_n + \frac{\gamma_b - 1}{2} \frac{M_1 \tau}{\rho_b} \delta \rho_n, \tag{51}
\]

FIG. 4. Same as Fig. 2, but for a pre-shock density ratio \( R_0 = 10^{-37} \) which corresponds to \( M_1 \approx 7 \times 10^{-6} \). The incident shock Mach number is \( M_1 \gg 1 \). The dimensionless tangential velocity is plotted as a function of \( A \) for three different dimensionless times: \( \tau = 1, 20, \) and 50.

B. Temporal evolution of the trailing edge perturbations

Yang et al.\(^{18} \) have found, as a result of their numerical calculations, that the trailing edge corrugations exhibit linear growth asymptotically in time. This result was later corroborated by Velikovich and Phillips,\(^{17} \) who obtained a series solution in powers of time to describe the ripples amplitude evolution. Kivity and Hanin also showed a similar behavior in the perturbation evolution of the trailing front perturbations (see their Fig. 6 in Ref. 20), but they did not make a deeper investigation of those results. This curious property of the rarefaction tail perturbations seems to have been dormant for almost 15 years until the numerical work of Yang et al. since the first results presented by Kivity and Hanin in Refs. 20 and 21. In this section we will solve the equation for the rarefaction tail ripples as a function of time. The solution is seen to be expressible in terms of known analytic functions and an explicit formula for the asymptotic growth rate is derived. We will limit our study to ideal gases with isentropic exponent \( \gamma_b < 3 \). The situations in which \( \gamma_b \geq 3 \) show certain features, the interpretation of which would require additional work that exceeds the scope of the present work. As is known from earlier works,\(^{17,18,20} \) the leading front ripples neither grow nor decrease in time, if the fluid ahead of the rarefaction wave is homogeneous. The perturbations that exist in the whole rarefaction fan [given, for example, as tangential velocity perturbations since \( t = 0^+ \), see Eq. (39)] never reach the rarefaction head. Therefore, there is no way to change the value of the rarefaction head ripples as there are no sound pressure fluctuations coming either from ahead or behind that front. This is not the situation for the rarefaction tail, which simply acts as a surface that transmits all the perturbations coming from the fluid layers in front of it. The perturbations that are generated inside the expansion wave make their way up to the rarefaction tail and are finally transmitted into fluid “\( b \)” to reach the contact surface at a later time. The rarefaction tail ripples evolve in order to ensure the continuity of total density and velocity at both sides of that front. Making use of the conservation laws at the position of the perturbed rarefaction tail it is possible to obtain the evolution equation for their amplitude. We will not repeat the details of the calculations here as they can be found elsewhere. The evolution equation for the trailing edge corrugations can be written as

\[
\tau \frac{d \tilde{u}_n}{d \tau} = \tilde{u}_n + \tau \delta u_n + \frac{\gamma_b - 1}{2} \frac{M_1 \tau}{\rho_b} \delta \rho_n, \tag{51}
\]
where \( \psi_n = \psi_n / \psi_0 \). The quantities \( \delta n_{\alpha} \) and \( \delta \rho_{\alpha} \) refer to the perturbed normal velocity and density at the rarefaction trailing edge. We use Eqs. (27) and (28) to get

\[
\frac{d \bar{\psi}_{\alpha}}{d \tau} = \frac{\bar{\psi}_{\alpha}}{\tau} + \frac{\gamma_b - 1}{2M_1} \left( \frac{\partial \varrho}{\partial \tau} \right)_{\alpha} - \frac{\gamma_b + 1}{2\tau} \left( \frac{\partial \varrho}{\partial \xi} \right)_{\alpha},
\]

(52)

where the partial derivatives are evaluated at the position of the rarefaction tail. Equation (52) is a linear inhomogeneous equation of first order. The partial derivatives form the inhomogeneous term. We can solve it by the method of variation of parameters.\(^{23}\) Care must be taken to integrate it since \( \tau = \tau_0 < 1 \), since the equation is singular at \( \tau = 0 \). At first, we compute the inhomogeneous term, which we call \( \text{inh}(\tau) \). To calculate the above partial derivatives, we make use of Eqs. (30), (31), and (44) after substituting for the corresponding values of \( \alpha = (\gamma_b + 1)/(\gamma_b - 1) \) and \( \beta = (\gamma_b - 3)/(\gamma_b - 1) \).

We decompose the inhomogeneity as

\[
\text{inh}(\tau) = \text{inh}_1(\tau) + \text{inh}_2(\tau) + \text{inh}_3(\tau),
\]

(53)

with

\[
\text{inh}_1(\tau) = \frac{3 - \gamma_b}{4n} \int_{\xi_n}^{\xi_n} \delta w_1(z) \sqrt{\xi_n - z} J_0(\tau \sqrt{nM_1^\beta(\xi_n - z)}) dz ,
\]

(54)

\[
\text{inh}_2(\tau) = -\frac{\psi_0^0}{\tau} + \frac{3 - \gamma_b}{4M_1 - a_1} \times \int_{\xi_n}^{\xi_n} \delta w_1(z) J_0(\tau \sqrt{nM_1^\beta(\xi_n - z)}) - 1\frac{1}{\tau} dz,
\]

(55)

\[
\text{inh}_3(\tau) = \frac{3 - \gamma_b}{4nM_1^\beta} \times \int_{\xi_n}^{\xi_n} \delta w_1(z) J_1(\tau \sqrt{nM_1^\beta(\xi_n - z)}) \frac{1}{\sqrt{\xi_n - z}} dz ,
\]

(56)

in which \( \xi_n = M_1^\beta, \) and \( \psi_0^0 \) is the corrugation of the trailing front at \( \tau = 0 + \). For the situations covered in this work, \( \xi_n > 1 \), because \( \gamma_b < 3 \). The functions \( J_0 \) and \( J_1 \) are the ordinary Bessel functions of zero and first order, respectively.

It is not difficult to see that the solution to Eq. (52) is given by

\[
\bar{\psi}_{\alpha}(\tau) = \bar{\psi}_{\alpha}^0 \tau + \tau \int_{\tau_0}^{\tau} \text{inh}(\tau) dt,
\]

(57)

where it is understood that \( \tau_0 \ll 1 \). At a first glimpse, the solution shown above seems to depend on the value chosen for \( \tau_0 \). However, substituting the function \( \text{inh}(\tau) \) above, we find after careful integration that the apparent dependence on \( \tau_0 \) disappears in the limit \( \tau_0 \rightarrow 0 \). In fact, the time integral can be done explicitly and we obtain the following formula for the trailing edge ripples:

\[
\bar{\psi}_{\alpha}(\tau) = \psi_0^0 \frac{\tau}{(1 + \frac{1}{2} M_1 \frac{\gamma_b - 1}{\gamma_b + 1})},
\]

FIG. 5. Dimensionless trailing edge ripples amplitude as a function of dimensionless time. The vertical axis is normalized with the initial interface ripple amplitude before shock (\( \psi_0 \)). The horizontal axis is \( ku_i t \) where \( u_i \) is the incident shock velocity. The plot compares well with the similar graphic in Ref. 18.

\[
\bar{\psi}_{\alpha}(\tau) = \psi_0^0 + \frac{3 - \gamma_b}{4M_1} \left[ \delta v_{\tau}(0 +) - \delta v_{\tau}(\tau) \right] + \tau \left[ \frac{3 - \gamma_b}{4} \right]
\]

\times M_1^{\beta} \int_{\xi_n}^{\xi_n} \delta w_1(z) J_1(\tau \sqrt{nM_1^\beta(\xi_n - z)}) \frac{1}{\sqrt{\xi_n - z}} dz,
\]

(58)

where \( \delta v_{\tau} \) is the tangential velocity perturbation at the trailing edge.

The function \( j_1(x) \) [it is a functional of the ordinary Bessel function \( J_1(x) \)] is given by\(^{27,28}\)

\[
j_1(x) = \int_0^x J_1(t) dt = \frac{1}{2} x J_2(1) + 3 x^2 - \frac{1}{4}
\]

(59)

The function \( _1F_2(a;b;c;x) \) is a generalized hypergeometric function whose series expansion is\(^{27,28}\)

\[
_1F_2(a;b;c;x) = \frac{(b) \Gamma(c)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(b+k) \Gamma(c+k)} \frac{x^k}{k!}.
\]

(60)

with \( \Gamma(x) \), the \( \Gamma \) function.\(^{27,28}\)

Inserting the series expansion above into Eq. (58) together with the series expansion for the Bessel function \( J_0 \) (Refs. 27, 28) would allow us to obtain a series expansion in powers of time for the trailing edge perturbation. The remaining integrals in \( z \) are easily computed with analytical expressions that can be systematically obtained.\(^{28}\) The complete series expansion is shown in the Appendix. In Fig. 5 we show the temporal evolution of the trailing edge ripples for the situation in which a shock coming from air (\( \gamma_{air} = 7/5 \)) impinges upon a surface separating it from He (\( \gamma_{He} = 5/3 \)). The incident shock Mach number is \( M_i = 1.24 \). The solution compares well with the numerical solution of Yang et al.\(^{18}\)

The horizontal axis is the dimensionless time defined by \( \tau_i = ku_i t \) (in units of the incident shock speed). The vertical axis is the ripple amplitude in units of \( \psi_0 \). After some gentle oscillations, the ripples reach a constant velocity.
IV. ASYMPTOTIC RATE OF GROWTH

A. Asymptotic growth rate as a function of the rarefaction strength

It is not difficult to see that the function $j_k \to 1$ asymptotically in time. Then, the last integral in Eq. (58) tends to a constant value for large times. This gives us a constant rate of growth for the trailing edge ripples. This is the result found in previous works.\textsuperscript{17,18} We have the results of the last section at our disposal to write an explicit analytic formula for the asymptotic growth rate. We make the definition $\chi_{n}^{\infty} = (d/d\tau) \tilde{\psi}_{n}$. Then, from the previous equations, we get the following expression:

$$\chi_{n}^{\infty} = \frac{3 - \gamma_{b}}{4} \xi_{n}^{\infty} \int_{1}^{\xi_{n}^{\infty}} \frac{\delta w_{1}(z)}{\sqrt{\xi_{n}^{\infty} - z}} \, dz,$$

(61)

where $\chi_{n}^{\infty}$ denotes the asymptotic value of $\chi_{n}(\tau)$.

The integral in the last formula can be expressed in terms of hypergeometric functions as will be shown later on. At first, we need an explicit expression for $\delta w_{1}$. It can be obtained by differentiating Eq. (45) with respect to its argument:

$$\delta w_{1}(z) = \sum_{j=0}^{2} \alpha_{j} \epsilon_{j} z^{\gamma_{j} - 1},$$

(62)

where the coefficients $\alpha_{j}$ and the exponents $\epsilon_{j}$ have been defined in Eqs. (46)–(50).

We define

$$I = \int_{1}^{\xi_{n}^{\infty}} \frac{\delta w_{1}(z)}{\sqrt{\xi_{n}^{\infty} - z}} \, dz,$$

(63)

which can be decomposed as

$$I = \sum_{j=0}^{2} \alpha_{j} \epsilon_{j} I_{j},$$

(64)

where the quantities $I_{j}$ are given by

$$I_{j} = \int_{1}^{\xi_{n}^{\infty}} z^{\epsilon_{j} - 1} \, dz.$$

(65)

After some algebra, the integrals $I_{j}$ for $j = 1, 2$ can be calculated analytically:\textsuperscript{28}

$$I_{j} = \frac{\sqrt{\pi} \Gamma(\epsilon_{j} + 1) \xi_{n}^{\infty}^{\epsilon_{j} - 1/2}}{\epsilon_{j} \sqrt{\xi_{n}^{\infty}}} \, \text{Erf}(\epsilon_{j} + 1/2) - \frac{1}{\epsilon_{j} \sqrt{\xi_{n}^{\infty}}} \, 2^{F_{1}\left(1/2, \epsilon_{j}; \epsilon_{j} + 1; 1/\xi_{n}^{\infty}\right)}.$$

(66)

The function $2^{F_{1}(a,b;c;x)}$ is the Gauss hypergeometric function, whose series expansion is

$$2^{F_{1}(a,b;c;x)} = \sum_{k=0}^{\infty} \frac{\Gamma(c)}{\Gamma(a) \Gamma(b)} \frac{\Gamma(a+k) \Gamma(b+k)}{\Gamma(c+k)} \frac{x^{k}}{k!}.$$

(67)

The hypergeometric function defined above is convergent in the unit circle when the condition $\text{Re}(a + b - c) < 0$ is satisfied, which is true in Eq. (66).

The integral with $j = 0$ is calculated:

$$\frac{3 - \gamma_{b}}{4} \xi_{n}^{\infty} \int_{1}^{\xi_{n}^{\infty}} \frac{\delta w_{1}(z)}{\sqrt{\xi_{n}^{\infty} - z}} \, dz,$$

$$I_{0} = \frac{2 \sqrt{\xi_{n}^{\infty}}}{\xi_{n}^{\infty}} \sqrt{1 - \frac{1}{\xi_{n}^{\infty}}}.$$

(68)

The growth rate is therefore calculated with the aid of the previous results and it can be written as

$$\chi_{n}^{\infty} = \frac{3 - \gamma_{b}}{4} \left( M_{1}^{\infty} \right)^{\gamma_{b} - 3/2} \sum_{j=0}^{2} \alpha_{j} \epsilon_{j} I_{j}.$$

(69)

For a given value of $R_{0}$, the value of $M_{1}$ decreases as the shock intensity increases. For a shock of infinite intensity, we would get the minimum possible value of $M_{1}$ for the set of given $\gamma_{a}$, $\gamma_{b}$, and $R_{0}$ values. This minimum value is the quantity $M_{1}^{\text{min}}$ defined in Sec. I [Eqs. (11) and (12)]. Because $M_{1}^{\text{min}}$ exhibits a wide range of variation for the density jumps chosen, we have found it convenient to plot the growth rate as a function of a rescaled rarefaction strength, defined as

$$\Sigma = \frac{M_{1} - M_{1}^{\text{min}}}{1 - M_{1}^{\text{min}}}.$$

(70)

In Fig. 6(a) we show the asymptotic ripple rate of growth for the gases with $\gamma_{a} = 4/3$, $\gamma_{b} = 6/5$, and different values of the pre-shock density ratio: $R_{0} = 7/10$, 3/10, and 1/10. The corresponding values of $M_{1}^{\text{min}}$ are: 0.9842, 0.9387, and 0.8793, respectively. We do the same in Fig. 6(b) but for $R_{0} = 1/100$ and 1/1000, and the corresponding values of $M_{1}^{\text{min}}$ are: 0.7581 and 0.6458, respectively. The asymptotic growth
The previous formulas are not true. For those values of $\gamma$ when
$\gamma = 5/3$, the corresponding integrals $I_2$ should be calculated
separately in such a way that the indeterminacy in the value of $\Gamma(\epsilon_j)$ does not appear in Eq. (66).

**Fluids for which either $\epsilon_1$ or $\epsilon_2$ is a negative integer**

The formulas and expansions of the last paragraph are valid as long as $\epsilon_j$ is not a negative integer. This means that there are two discrete sets of infinite values of $\gamma$, for which the previous formulas are not true. For those values of $\gamma$, the corresponding integrals $I_1$ should be calculated separately. The sets of problematic $\gamma$ values can be easily calculated and are given by

\[
\gamma = \begin{cases} 
\frac{6m-1}{2m+1} & m=1,2,3,\ldots \\
\frac{6m+1}{2m+3} & m=1,2,3,\ldots 
\end{cases} \quad \begin{cases} 
\frac{5}{3}, \frac{11}{5}, \frac{17}{7}, \ldots \\
\frac{7}{5}, \frac{13}{7}, \frac{19}{9}, \ldots 
\end{cases},
\]

(71)

when $\epsilon_1 = -m$, or

\[
\gamma = \begin{cases} 
\frac{6m-1}{2m+1} & m=1,2,3,\ldots \\
\frac{6m+1}{2m+3} & m=1,2,3,\ldots 
\end{cases} \quad \begin{cases} 
\frac{5}{3}, \frac{11}{5}, \frac{17}{7}, \ldots \\
\frac{7}{5}, \frac{13}{7}, \frac{19}{9}, \ldots 
\end{cases},
\]

(72)

when $\epsilon_2 = -m$.

We recognize that $\gamma = 5/3$ and $\gamma = 7/5$ correspond to one or the other set of coefficients. To study a specific case and see how the calculations proceed in the general case, let us consider the case $\gamma = 5/3$. For this value of the adiabatic index, it is $I_1$ that must be calculated separately, because $\epsilon_1 = -1$. After some algebra, it can be calculated analytically, by using a mathematical software:

\[
I_1 = \frac{\sqrt{\xi_n - 1}}{\xi_n} + \frac{1}{\xi_n^{3/2}} \ln \left[ \sqrt{\xi_n + \sqrt{\xi_n - 1}} \right].
\]

(73)

This is the value of $I_1$ that must be used to calculate the growth rate in the whole range of $M_1$ values in Eq. (69). For this value of $\gamma$, $I_2$ can still be inferred from Eq. (66).

In Fig. 7(a) we plot the asymptotic rate of growth [Eq. (69)] as a function of the rescaled rarefaction strength [Eq. (70)], for $\gamma = 5/3$, $\gamma = 5/3$, and different values of the pre-shock density ratio: $R_0 = 7/10$, $3/10$, and $1/10$. The values of $M_{1\text{min}}$ are in these cases: 0.9620, 0.8703, and 0.7525, respectively. In Fig. 7(b) we do the same for $R_0 = 1/100$ and $1/1000$. For these density ratios, the minimum values $M_{1\text{min}}$ are 0.5350 and 0.3585, respectively.

We see that $\gamma = 7/5$ is also one of the troublesome coefficients. In this case, it is $I_2$ which must be calculated separately and not with Eq. (66). It can be seen that $I_2(5/3) = I_2(7/5)$ because $\epsilon_1(5/3) = \epsilon_2(7/5) = -1$. Therefore, only a minor change would be necessary in Eq. (69) to study this other combination.

**B. Asymptotic growth rate in the limits of strong and weak rarefactions**

Let us suppose for the moment, that the density of fluid “a” is orders of magnitude lower than that of fluid “b.” This is a common situation met in ICF experiments, in which the inner layer of the target borders almost with vacuum. Any shock wave that hits this interface will generate a strong rarefaction going backwards toward the ablation surface. Any corrugation initially present in the interface that forms the rear side of the target will be carried away by the corrugated expansion wave, generating pressure and velocity disturbances which would affect the mass distribution inside the target. This process, known as “feed-out” is certainly of importance for the design of a successful implosion experiment.5–7,24–26 It becomes clear that any piece of knowledge about the behavior of the perturbations inside the rarefaction, for example in feed-out conditions, would be useful as a guide in the design of future numerical or real experiments. Therefore, it is convenient to do a mathematical study of the formulas deduced in the previous paragraphs, in the limit of a very strong rarefaction. We present, at first, an approximate formula that is valid when $M_1 \to 0$ (strong rarefaction), which is deduced from the exact analytical expression of Eq. (69). We also do the same analysis for the opposite case of a weak rarefaction. It is interesting to see how the growth rate scales with $M_1$ in both physical limits.

**1. Strong rarefaction**

In this case, the fluid “a” is very light compared to fluid $b$, that is: $p_{a0} \ll p_{b0}$. This means, mathematically speaking, that we are in the limit $M_1 \to 0$. For the case in which $R_0$
\[ I_0 = \frac{2}{\sqrt{\xi_1}} - \frac{1}{4} \frac{\xi_1^{3/2}}{\xi_1^{3/2}} - \frac{1}{8} \frac{\xi_1^{3/2}}{\xi_1^{3/2}} \frac{5}{64} \xi_1^{3/2} + O \left( \frac{1}{\xi_1^{3/2}} \right). \]

In order to deal with \( I_1 \) and \( I_2 \) we assume that \( \epsilon_1 \) is not a negative integer. Then, using Eqs. (66) and (67) in the limit \( \xi_1 \rightarrow 0 \), we get

\[ I_j \approx \frac{\pi \Gamma(\epsilon_j) \xi_1^{\epsilon_j - 1/2}}{\Gamma(\epsilon_j + 1/2)} - \frac{1}{\epsilon_j} \frac{\xi_1^{\epsilon_j - 1/2}}{\xi_1^{\epsilon_j - 1/2}} - \frac{1}{2(\epsilon_j + 1)} \frac{\xi_1^{\epsilon_j - 1/2}}{\xi_1^{\epsilon_j - 1/2}} \]

\[ - \frac{3}{8}\left(\epsilon_j + 2\right) \frac{\xi_1^{\epsilon_j - 1/2}}{\xi_1^{\epsilon_j - 1/2}} - \frac{5}{16(\epsilon_j + 3)} \frac{\xi_1^{\epsilon_j - 1/2}}{\xi_1^{\epsilon_j - 1/2}} - \frac{35}{128(\epsilon_j + 4)} \frac{\xi_1^{\epsilon_j - 1/2}}{\xi_1^{\epsilon_j - 1/2}} + \cdots. \]

After substituting the limiting expressions obtained for \( I_j \), we get

\[ \chi_\infty = \frac{3 - \gamma_b}{4} \sqrt{n(\alpha_0 + \alpha_1 + \alpha_2)} \sqrt{\xi_1^{3/2} + \alpha_1 \epsilon_1} \frac{\gamma_b - 3}{4} \sqrt{\frac{n\Gamma(\epsilon_1)}{\Gamma(\epsilon_1 + 1/2)}} M_1^2 + \frac{\gamma_b - 3}{4} \sqrt{\frac{n\Gamma(\epsilon_2)}{\Gamma(\epsilon_2 + 1/2)}} M_1 + \frac{\gamma_b - 3}{4} \sqrt{\frac{n\Gamma(\epsilon_3)}{\Gamma(\epsilon_3 + 1/2)}} M_1 + \gamma_b - 3 \]

\[ \times \sqrt{n} \left( \frac{\alpha_0}{2} - \frac{\alpha_1 \epsilon_1}{2(\epsilon_1 + 1)} - \frac{\alpha_2 \epsilon_2}{2(\epsilon_2 + 1)} \right) M_1^{(3 - \gamma_b)/2(\gamma_b - 1))} + \frac{\gamma_b - 3}{4} \sqrt{n} \left( \frac{\alpha_0}{8} - \frac{3\alpha_1 \epsilon_1}{8(\epsilon_1 + 2)} - \frac{3\alpha_2 \epsilon_2}{8(\epsilon_2 + 2)} \right) M_1^{(3 - \gamma_b)/2(\gamma_b - 1)} \]

\[ + \frac{\gamma_b - 3}{4} \sqrt{n} \left( \frac{5\alpha_0}{128} - \frac{35\alpha_1 \epsilon_1}{128(\epsilon_1 + 4)} - \frac{35\alpha_2 \epsilon_2}{128(\epsilon_2 + 4)} \right) M_1^{(3 - \gamma_b)/2(\gamma_b - 1)} + \cdots. \]
The authors of Ref. 17 also deduced this scaling law for small $M_1$ values, the growth rate tends to zero as $\gamma_b = 5/3$. In any case, we just confirm their prediction. For other values of $\gamma_b$ which satisfy the condition $\epsilon_1$ equal to a negative integer, the specific scaling law must be calculated accordingly, as we have just done for $\gamma_b = 5/3$. However, it is not necessarily implied that the ripple velocity will scale as in the case $\gamma_b = 5/3$.

In Fig. 10 we show the exact growth rate [Eq. (69)] as a function of $M_1$ for a very small value of $M_1^{\text{min}}$. We have chosen $R_0 = 10^{-15}$, which gives us $M_1^{\text{min}} = 0.0016$, for $\gamma_a = \gamma_b = 5/3$. In Fig. 11, the solid line is the growth rate calculated with the complete formula [Eq. (69)] and the dotted line is the approximate value retaining the first two terms in the last expansion. If we compare Fig. 10 with Fig. 6 of Ref. 17, we see that both plots agree quite well in the low $M_1$ region, near $M_1 = 0$. For the weak rarefaction zone, our formula predicts higher values. The reason for the disagreement at higher $M_1$ values lies in the fact that we vary $M_1$ by changing the incident shock intensity at fixed pre-shock density ratio. In Ref. 17, the authors varied $R_0$ after having chosen a very strong incident shock. In our curve in Fig. 10, the shocks corresponding to points with $M_1 = 0.1$ are quite strong, and therefore the agreement is reasonable there, despite the difference discussed.

### 3. Weak rarefactions

When the density jump at the material interface is near unity, the fluid $b$ does not expand so much as in the case of very small values of $R_0$. This means that the rarefaction strength will also be near 1. In this case, it is possible to find an approximate scaling law, valid for $M_1 \rightarrow 1$. The main result is that the asymptotic growth rate scales as $\sqrt{1 - M_1}$, and the proportionality constant depends on $\gamma_b$ and $\gamma_a$. To get the scaling law in this limit, it is useful to re-write the integral $I_j$ in the following way:

$$I_j = 2M_1^{\beta(\epsilon_j - 1/2)} \sqrt{1 - \frac{1 - \epsilon_j}{\epsilon_1} F_1 \left(1 - \epsilon_j, \frac{1}{2}; \frac{3}{2}; \frac{1}{\epsilon_1} - 1\right)}.$$  

(79)
The hypergeometric function defined above does not satisfy the criterion for convergence on the unit circle. However, we are only interested in the limit in which \( M_1 \rightarrow 1 \), which is equivalent to \( \xi_1 \rightarrow 1 \), and therefore the series can be well approximated by the first few terms. Indeed, we only retain the first term which equals unity. Then, the only task is to approximate the square root. We get

\[
I_0, I_{\infty} = 2 \sqrt{\frac{3 - \gamma_b}{\gamma_b + 1} \sqrt{1 - M_1}},
\]

because \( I_0 \) has the same scaling near \( M_1 \approx 1 \).

Substituting these scalings inside Eq. (69), we can write

\[
\chi_{\infty} = \frac{1}{2} \sum_{k=0}^{\infty} \left( (\gamma_b - 1)k \frac{\gamma_b - 3}{2} \alpha_k \right) \sqrt{\frac{\gamma_b + 1}{\gamma_b - 1} \sqrt{1 - M_1}} + O(1 - M_1)^{3/2},
\]

which is the generalization of the scaling law obtained in Ref. 17, to arbitrary values of the initial fluids parameters. We compare in Fig. 12 the previous approximate expression with the complete formula for the case in which \( \gamma_a = 4/3 \), \( \gamma_b = 6/5 \) and three different initial density ratios: \( R_0 = 7/10 \), 1/10, and 1/100. The corresponding values of \( M_{1,\text{min}} \) are 0.9842, 0.8793, and 0.7581, respectively. We can see that the approximate scaling law reproduces better the growth rate for \( M_1 \) values near unity. The agreement tends to be even better, obviously, for \( R_0 \) values near 1, as in those cases it is \( M_{1,\text{min}} \) also near unity.

V. QUALITATIVE DISCUSSION

So far, we have presented in the previous sections an analytical model that allows us to calculate the perturbation profiles inside a rarefaction wave. In particular, we have focused our attention to get an exact analytically closed expression for the rarefaction tail ripples. In Ref. 18, the authors showed that those ripples evolve linearly at large times. Soon after, an analytical model confirmed those findings.

In it, the authors expanded each magnitude inside the rarefacting fluid as a Taylor series in powers of time and could follow the temporal evolution of the perturbed quantities. The drawback of such a method is that it is not always easy or possible to obtain closed analytical expressions for the quantities of interest. The authors could only show an analytical scaling law for the tail ripples in the limit of weak rarefactions, valid for strong incident shocks. On the contrary, the model presented here gives exact and closed analytical expressions for all the magnitudes developing inside the expanding fluid and in particular for the perturbations growing at its trailing edge.

One possible application of the results shown here could certainly be their use in the benchmarking of hydrodynamic codes in two-dimensional simulations in which shock induced perturbations are the key issue. This is the typical environment that deals with the RM instability problem. Besides, disposing of an exact analytical model should also be helpful for gaining a qualitative understanding of the hidden mechanisms that could possibly drive the asymptotic evolution of the perturbations. In particular, for the present problem, one of the characteristics which seems difficult to explain in qualitative terms is the asymptotic linear growth observed at the trailing edge. Neither Ref. 17 nor Ref. 18 have given a simple explanation of that fact. Unfortunately, taking a look at our Eq. (58), it does not seem to be of much help either in clarifying the picture. The same can be certainly said for the RM instability problem, in which an exact expression for its growth rate at the contact surface is also known. However, the reason the interface grows linearly in time is hidden in the subtleties of the interaction between the interface circulation and the deformed fronts by means of traveling pressure fluctuations. As time passes, the density disturbances associated with the sound waves will vanish and the fluids velocity perturbations will behave as an incompressible field giving rise to a constant normal perturbation velocity at the interface. Its exact value will depend on the complete interaction history between the interface and the deformed wavefronts. The situation seems to be even more puzzling for the rarefaction problem studied here, as we are considering a quite innocuous environment: neither vorticity nor entropy disturbances exist, and there is also no gravity driving the flow as, for example, in the RT problem.

Anyway, we will try in this section to get a simpler picture of the asymptotic behavior of the trailing edge ripples. In order to do so, we reformulate a little bit the mathematical approach, starting from the same conservation equations that have been solved before. We will adapt the known Riemann invariants theory in one dimensional (1-D) to this essentially two-dimensional (2-D) situation, but in an approximate way. To do it clearly, we write again the mass and \( x \) momentum conservation equations, much in the same way as was done in Ref. 16, pages 15 to 20. However, we include now the tangential velocity, which is not present in the one-dimensional problem. As usual, we start with the complete equations of motion, without linearizing them, for the moment. At first, we recall that the flow inside the rarefaction is adiabatic:

\[
\frac{d\rho}{dt} = \frac{1}{c^2} \frac{dp}{dt},
\]

where \( \rho \) is the fluid density, \( c \) is the adiabatic sound speed, and \( p \) is the fluid pressure. The operator \( d/dt \) is intended to
mean the total time derivative following a particle path \((d/dt = \partial \partial t + \mathbf{v} \cdot \mathbf{V})\). Substituting the above expression inside the mass conservation equation [Eq. (20)], we can write in 2-D, after some algebra:

\[
\frac{1}{\rho c} \frac{\partial \mathbf{v}}{\partial t} + \frac{\mathbf{v} \cdot \partial \mathbf{v}}{\partial x} + \frac{\mathbf{v} \cdot \partial \mathbf{v}}{\partial y} + c \frac{\partial \mathbf{v}}{\partial x} + \frac{c \partial \mathbf{v}}{\partial y} = 0,
\]

(83)

where \(v_x, v_y\) are the normal and tangential velocity components, respectively.

The momentum conservation equation in the \(x\) direction reads

\[
\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + \frac{1}{\rho} \frac{\partial \rho}{\partial x} = 0.
\]

(84)

Subtracting and adding one from another, and later re-arranging, we get

\[
\frac{\partial v_x}{\partial t} + (v_x - c) \frac{\partial v_x}{\partial x} - \frac{1}{\rho c} \left[ \frac{\partial \rho}{\partial t} + (v_x - c) \frac{\partial \rho}{\partial x} \right] = c \frac{\partial v_y}{\partial y} - v_y \frac{\partial v_y}{\partial y} + v_x \frac{\partial v_y}{\partial y},
\]

(85)

\[
\frac{\partial v_x}{\partial t} + (v_x + c) \frac{\partial v_x}{\partial x} + \frac{1}{\rho c} \left[ \frac{\partial \rho}{\partial t} + (v_x + c) \frac{\partial \rho}{\partial x} \right] = -c \frac{\partial v_y}{\partial y} + v_x \frac{\partial v_y}{\partial y} - v_y \frac{\partial v_y}{\partial y}.
\]

(86)

As it stands, it is formally correct, and nothing seems to be gained from this rearrangement of the conservation equations which we have already solved, in the linear approximation, in the previous sections. However, as is done in 1-D theory, let us re-interpret in an approximate way the quantities on the left-hand sides as derivatives along the 1-D \(C_\pm\) and \(C_+\) characteristics in the \((x,t)\) plane. Let us define the quantities \(J_\pm\) and \(J_+\), such that their variations are given by

\[
dJ_\pm = dv_x \pm \frac{dp}{\rho c},
\]

(87)

which in 1-D theory would coincide with the \(J_-\) and \(J_+\) Riemann invariants. We note that Eqs. (85) and (86) can be rewritten as

\[
\left( \frac{d J_-}{dt} \right)_{C_-} = c \frac{\partial v_x}{\partial y} - v_x \frac{\partial v_x}{\partial y} + v_y \frac{\partial v_x}{\partial y},
\]

(88)

\[
\left( \frac{d J_+}{dt} \right)_{C_+} = -c \frac{\partial v_x}{\partial y} + v_x \frac{\partial v_x}{\partial y} - v_y \frac{\partial v_x}{\partial y},
\]

(89)

where the derivatives have to be calculated in the \((x,t)\) plane along the paths defined by

\[
C_\pm : \frac{dx}{dt} = v_x \pm c.
\]

(90)

Rigorously speaking, the time derivatives following a surface in the \((x,y,t)\) space should also include convective derivatives along the \(y\) direction in Eqs. (88) and (89). However, those terms would vanish in linear theory, and are therefore not included in a first approximation. Furthermore, also the second and third terms on the right-hand side of that equation give no contribution in the linear theory and will be not considered in what follows. In a strictly 1-D problem, the quantities \(J_-\) and \(J_+\) would be conserved along the \(C_\pm\) trajectories defined above. In our case, however, we find that their value is modified along the characteristics, and the way this happens will depend on the different values of the tangential flow velocity along them, as will be seen below. For an ideal gas, we can use the expressions given in Ref. 16:

\[
J_\pm = v_x \pm \frac{2}{\gamma_b - 1}.
\]

(91)

We are interested in the linear versions of Eqs. (88)–(91), and therefore we linearize them to obtain the simpler expressions:

\[
\frac{d d \delta J_-}{dt} = kc \delta v \cos ky,
\]

(92)

\[
\frac{d d \delta J_+}{dt} = -kc \delta v \cos ky,
\]

(93)

where use has been made of the facts that the tangential velocity is given by \(\delta v_\pm = \delta v(x,t) \sin ky\) and that the zero-order normal velocity and pressure do not depend on the tangential coordinate, in linear theory. It is seen that the main difference with a strictly 1-D problem is that \(\delta J_-\) and \(\delta J_+\) change along the characteristic lines because of the lateral mass flow, an ingredient which is absent in the non-perturbed problem. We evaluate the last equations at the rarefaction trailing edge, first noting that the total perturbation of the quantity \(J_-\) is evaluated at the moving front, and therefore, the convective contribution should also be added: \(17,18\)

\[
\frac{d}{dt} (\delta J_-) + \frac{d}{dt} \left( \frac{d J^0_-}{dx} \psi_n \right) = kc_n \delta v.
\]

(94)

\[
\frac{d}{dt} (\delta J_+) + \frac{d}{dt} \left( \frac{d J^0_+}{dx} \psi_n \right) = -kc_n \delta v.
\]

(95)

We have also dropped the \(\cos ky\), for simplicity, on the understanding that both sides depend in the same way on the tangential coordinate. The quantities \(J^0_-\) and \(J^0_+\) are the 1-D Riemann invariants of the unperturbed problem, shown before.\(^17\) From the model developed in the previous sections, it is easy to see that \(\delta J_-\) and \(\delta J_+\) can be written as functions of \(\delta v\) and that both quantities will decay to zero for large times. Then, asymptotically, we expect the following equalities to hold at the rarefaction tail:

\[
\frac{1}{c_{b1}} \frac{d}{dt} \frac{d J^0_-}{dx} \psi_n \bigg/ t \approx 0,
\]

(96)

\[
\frac{1}{c_{b1}} \frac{d}{dt} \frac{d J^0_+}{dx} \psi_n \bigg/ t \approx 0.
\]

(97)

Equation (96) actually gives us no information in linear theory about the rarefaction trailing edge ripple, because for a right facing rarefaction, the invariant \(J^0_+\) is uniform in the whole space, in the unperturbed problem. That is, Eq. (96) is satisfied trivially. However, Eq. (97) gives non-trivial infor-
mation because \( J_+^0 \) has non-zero derivative. We see that the only way to satisfy it is by allowing \( \rho_n \) to grow linearly in time. The time coordinate in the denominator of the left-hand side of the above equations is a consequence of the self-similarity of the 1-D unperturbed flow. Physically, when we study the problem for large times, the perturbations \( \delta \nu \), \( \delta \theta \), \( \delta \rho \), etc., inside the rarefaction region will vanish. Equivalently, we could say that for large times, the fluid profiles for velocity, density, etc., would resemble more and more the 1-D profiles with no perturbations, and therefore, they would depend on \( x \) and \( t \) through the similar self-similarity \( x/\tau \).

This means that the quantity \( (x_n(t) + \psi_n(t))/\tau \) should be time-independent, which is only possible if \( \psi_n \propto t \), asymptotically in time. The asymptotic equivalence with the 1-D problem can only be approximate, because, in the 2-D situation the leading and trailing fronts are distorted, while in the 1-D problem they are planar. The only fact we can deduce from Eqs. (92)–(97) is that the corrugation at the trailing front continues increasing linearly in time just to ensure that the total value of \( J_+ \) reaches a constant value on that front.

We can see these results from another angle. Let us go back to Eq. (88) and evaluate it at a large time \( t \). Let us assume a small time increment \( \Delta t \ll t \). Then, if we neglect quantities like \( \delta \nu \) and functions of it, Eq. (89) can be approximately written as

\[
J_+^0 (x_n(t+\Delta t) + \psi_n(t+\Delta t)) = J_+^0 (x_n(t) + \psi_n(t)), \tag{98}
\]

where \( x_n^+ = c_n M_1 t \). After a straightforward Taylor expansion, the last equation is seen to be equivalent, because of the self-similarity of the 1-D problem, to the following condition:

\[
\frac{\psi_n(t+\Delta t)}{t+\Delta t} \approx \frac{\psi_n(t)}{t}, \tag{99}
\]

which implies that \( \psi_n \propto t \). Physically, there is an asymptotic constant excess value for \( J_+ \) at the crest of the trailing edge corrugation. This implies that there will be an asymptotic constant perturbation value for the pressure on that position.\(^{17}\) Similar results were deduced in Ref. 17 for the total pressure and normal velocity perturbations at the tail of the rarefaction when deriving the evolution equation for the tail ripples.

To sum up, as for the observed asymptotic linear growth of the rarefaction tail ripples, it seems to be the consequence of the following facts: at first, as the fluid ahead of the rarefaction is irrotational and isentropic, the perturbations inside the rarefaction will vanish everywhere for large times, and then, the quantity \( J_+ \) at the rarefaction trailing front remembers its previous values for large times. That is, \( J_+ \) tends to a constant at the rarefaction tail, and because that front is already distorted, the late time constancy of \( J_+ \) there implies that the ripple should grow linearly in time, because of the self-similarity of the background pressure, density, and normal velocity profiles.

**VI. SUMMARY**

We have presented an analytical model to calculate the temporal evolution of the trailing edge ripples in a corrugated rarefaction. The model applies to rarefactions generated by the collision of a planar shock wave against a corrugated material interface that separates two different fluids. The ripple amplitude at the rarefaction tail grows linearly in time for very large times after some damped oscillations due to the pressure field generated inside the rarefaction. Previous numerical work is seen to agree exactly with the predictions of the model. The asymptotic growth rate is studied as a function of the rarefaction strength, for given pre-shock parameters, namely: initial density ratio at the contact surface, incident shock intensity and isentropic coefficients of both fluids. The study has been limited only to expanding gases which have \( \gamma < 3 \). It has been found that the growth rate can be expressed in closed form. There are certain values of the isentropic coefficient which require a separate mathematical treatment, which, however, does not preclude obtaining an analytic result. The approximate formula for the limiting transition of a strong rarefaction has been obtained. For a continuum set of \( \gamma \) values, the growth rate is seen to decay to zero linearly with \( M_1 \) if \( \gamma < 5/3 \). If \( \gamma = 5/3 \) it decays as \( M_1 \sqrt{\pi M_1} \), and if \( \gamma > 5/3 \) it goes to zero as \( M_1^{(3-\gamma)/(2(\gamma-1))} \). The known approximate formula in the limit of very weak rarefactions has been generalized for any value of the pre-shock parameters. Also, a semi-qualitative and approximate explanation of the asymptotic linear growth at the rarefaction tail has been tried. The exact expressions presented in this work could be useful as a theoretical guide in the design of future real or numerical experiments involving rarefactions induced by shock waves.

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**APPENDIX: SERIES EXPANSION OF THE RIPPLE AMPLITUDE IN POWERS OF TIME**

We show the expansion in powers of time of Eq. (58):

\[
\bar{\psi}_n = \bar{\psi}_n^0 + \frac{3-\gamma_b}{4} \delta \nu_n^0 + \frac{\gamma_b-3}{2} \sum_{k=0}^{\infty} \left( -M_1^{k} \right) ^{2} + \frac{2}{(2k+1)(k+1)} \left( \frac{nM_1^q}{4} \right) ^{k} \left( -1 \right) ^{k} D_\kappa \tau ^{2k}, \tag{A1}
\]

where

\[
D_\kappa = \sum_{j=0}^{2} \alpha_j \epsilon_j d_{jk}, \tag{A2}
\]

and the coefficients \( d_{jk} \) are given by

\[
d_{jk} = \sum_{j=0}^{k} \frac{k!}{(k-j)!} \frac{\xi_{jk}^{k-j}}{\epsilon_j + 1} \left( \epsilon_j + 1 \right) \left( \xi_{jk}^{k-j} - 1 \right). \tag{A3}
\]
If \( e_j = -m \), with \( m = 0, 1, 2, \ldots \), the corresponding term inside the sum that defines the coefficient \( d_{jk} \) must be substituted by (that is, the term with \( l = m \))

\[
\frac{k!}{(k-m)!m!} \xi_{rt}^{k-m} \ln \xi_{rt}.
\]  

(A4)