Asymptotic freeze-out of the perturbations generated inside a corrugated rarefaction wave

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(Received 18 March 2004; accepted 28 May 2004; published 9 August 2004)

Based on previous work [J. G. Wouchuk and R. Carretero, Phys. Plasmas 10, 4237 (2003)], the conditions of asymptotic freeze-out of the ripples at the tail of a corrugated rarefaction wave are analyzed. The precise location of the freezing-out regions in the space of preshock parameters is tried, and an efficient algorithm for their determination is given. It is seen that asymptotic freeze-out can only happen for gases that have an isentropic exponent $\gamma < \gamma_c = 2.2913$. It is shown that the late time freeze-out of the ripple perturbations is correlated to the initial tangential velocity profile (at $t=0^+$) inside the expansion fan. © 2004 American Institute of Physics.

[DOI: 10.1063/1.1775011]

I. INTRODUCTION

The evolution of the perturbations generated by corrugated shocks or rarefaction waves pertains to one of the fundamental problems in the study of hydrodynamic instabilities.\(^1\)-\(^{28}\) In particular, it is of main interest in diverse fields as astrophysics, shock tube research, and inertial confinement fusion (ICF). Despite that the study of the stability of shock waves has received considerable attention in the last 50 years, very little has been done to study, with as much detail, the stability properties of corrugated simple expansion waves.\(^9\) In particular for ICF research, the perturbations evolving inside the rarefactions generated after the unloading of the first sequence of shocks on the rear surface of a typical target, will later have an influence in the evolution of the more important Rayleigh-Taylor instability (RTI) during the ablative acceleration phase.\(^5\)-\(^7\),\(^10\)-\(^12\),\(^14\)-\(^17\)

Quite recently, an explicit analytic expression for the tail ripple evolution, valid for any values of the preshock parameters, has been obtained.\(^26\) Therefore, the complete compressible history and the asymptotic rate of growth of the trailing edge ripples can be predicted with a closed analytical formula. In this work we will concentrate on the conditions to be met in order for that asymptotic velocity to be zero, that is, the freeze-out conditions at the rarefaction tail. These studies are motivated by two principal reasons, which are discussed in the following.

As is already known after many years of scientific work devoted to the study of hydrodynamic instabilities in ICF environments, the most dangerous instability in this context is the RTI. It is also known that the perturbations left by the corrugated shocks or rarefactions which exist prior to the acceleration stage, play an important part too. In fact, the perturbations generated by shocks or rarefactions serve as seeds for the later RTI. The research in this field evolves principally in two interdependent fronts: experiment and simulation. Because of this, any analytical studies on the subject may become of great value in the validation and benchmarking of the hydrodynamic codes that are commonly used in this type of research. The results presented in the following sections are based on rigorous and exact analytical linear theory, and therefore, they can be very helpful in the design of more accurate real or numerical experiments aimed toward a deeper understanding of the basic physics that drives hydrodynamic instability growth. This is even more true when difficult features of the problem, like compressibility, play an important role in the time evolution of the perturbation field.\(^5\)-\(^7\),\(^17\)

On the other hand, referring to Fig. 1, we recognize the basic scenario that develops when an incident shock hits a contact surface (at $t=0$) and a rarefaction is reflected backwards. In an ICF target, for example, the picture shown is known to be valid before the rarefaction arrives to the ablation surface. The contact surface corrugation starts to grow for $t>0+$ and this is the well known Richtmyer-Meshkov instability (RMI).\(^3\)-\(^{25}\) The interface ripple starts oscillating at first due to the sound waves reverberating at both sides on each fluid and later achieves a constant rate of growth in linear theory. The calculation of the asymptotic linear growth at the contact surface has been quite extensively reviewed in the recent past and will not be considered here.\(^4\)-\(^{25}\) We just mention that the physical mechanism that drives growth at the contact surface is different from the mechanism driving the growth at the rarefaction tail.\(^20\),\(^21\),\(^26\) Besides, growth at the contact surface can also show freeze-out.\(^6\),\(^26\)\(^,\)^\(^27\) Despite the fact that the growth at the rarefaction tail is not influenced by the growth at the contact surface, the reverse is not true. Therefore, if there is freeze-out at the tail of the rarefaction, this fact would somehow control the freezing-out at the contact surface. Therefore, a causal relationship between both situations of freeze-out is not to be excluded a priori.

There is some evidence in the recent literature that shows the possibility of having freeze-out at the contact surface, even for monatomic ideal gases at both sides of the material interface.\(^6\) Therefore, to better understand the mechanisms that could drive the freeze-out of the RMI at the contact surface, it is better to attack the freeze-out at the rarefaction tail at first. An analytical procedure to determine the regions of freeze-out at the trailing edge is therefore the main purpose of the present work.

Taking into account these facts, it is hoped that the
present work could be useful to fulfill at least two general objectives. At first, these results could be of great value if coupled to the numerical studies currently underway on the seeds of the RTI in the ablative environment of ICF targets. And second, the results presented here could also be quite useful in order to understand the more difficult freeze-out problem for the rarefaction reflected RMI, in situations in which the gas to the left side of the rear surface has a very low density. Of course, these last two objectives have to be restricted to gases with \( \gamma_b, \gamma_y < 3 \). The zero order (unperturbed profiles) can be calculated with expressions already developed in previous works, and will not be repeated here. The dimensionless self-similar coordinates of the rarefaction fronts are \( \zeta_r = M_1 t \) and \( \zeta_y = 1 + (v_i - v_1) / c_{bf} \). In this subsection we consider the time evolution of the perturbations generated inside the rarefaction due to the small corrugation imposed at the contact surface. We define the dimensionless time \( \tau = k c_{bf} t \). We will not solve the linearized fluid equations in any detail here, as this has already been done in a previous work. Rather, we will use those results and present the equations that govern the temporal evolution of the ripples imposed at the tail of the rarefaction. It has been seen that a useful quantity is the initial tangential velocity profile. Let us write the tangential velocity at any time and position in the form

\[
\delta v(x, y, t) = c_{bf} k \psi_0 \delta v(x, t) \sin k y.
\]

The tangential velocity profile at \( t = 0^+ \) is given by

\[
\delta v(\zeta, t = 0^+) = \left[ \frac{1}{\gamma_y + 1} \frac{c_{bf}}{u_i} \left( \zeta^2 - \zeta_{rh}^2 \right) + \frac{2}{\gamma_y + 1} \left( \frac{\psi_0}{u_i} - c_{bf} \right) \right] \times (\zeta - \zeta_{rh}).
\]  

II. BASIC EQUATIONS

A. Perturbed profiles

In Fig. 1 we show a typical scheme for the generation of a corrugated rarefaction wave and the transmitted shock. A planar incident shock (not shown) has come from the left in the laboratory system, moving with velocity \(-u_i \hat{x}\). The contact surface, which separates two different fluids, is located at \( x = 0 \). The fluids have initial densities \( \rho_{a0} \) to the left and \( \rho_{b0} \) to the right. The incident shock compresses fluid \( b \) to the density \( \rho_{b1} \). The fluid velocity behind the incident shock was \(-v_1 \hat{x}\) in the laboratory frame. The contact surface has an initial corrugation with wavelength \( \lambda \) and initial ripple \( \psi_0 \). We assume from now on that \( \psi_0 \ll \lambda \), and therefore only a linear theory is used to study perturbation evolution. The initial interface ripple has the form \( \psi_0 \cos ky \), where \( k = 2\pi/\lambda \) is the perturbation wave number and \( y \) is the tangential coordinate, along the contact discontinuity. By the time the incident shock has been completely refracted at the contact surface, a corrugated expansion fan has been formed in fluid \( b \) and a transmitted shock has been generated in fluid \( a \). This is supposed to happen in a very small time interval around \( t = 0^+ \). For \( t > 0^+ \), we work in a system of reference comoving with the contact surface. The expansion region is delimited by the rarefaction leading and trailing edges, which move with the local sound speeds. In the contact surface frame of reference, the leading front moves to the right with velocity \(+c_{bf} + v_1 - v_1 \hat{x}\) and the trailing front moves with velocity \( c_{bf} \hat{x} \). The shock transmitted into fluid \( a \) moves to the left with velocity \(-u_i - v_1 \hat{x}\). The fluid density behind the transmitted shock is \( \rho_{a1} \). The density ahead of the rarefaction leading front is \( \rho_{b1} \), and its sound velocity is \( c_{bf} \). The density and sound speed at the trailing edge of the rarefaction wave are, respectively, \( \rho_{bf} \) and \( c_{bf} \). It is convenient to characterize the rarefaction strength with the ratio of sound speeds at the tail and the head of the rarefaction, as proposed in Ref. 20, and define the rarefaction Mach number: \( M_1 = c_{bf} / c_{b1} \). We will work with ideal gases that are characterized by the corresponding isentropic exponents (ratio of specific heats) \( \gamma_a \) and \( \gamma_y \), and restrict the discussion to gases with \( \gamma_y < 3 \).
The linearized equations of motion can be integrated to give us a constant integral in Ref. 26 has now been corrected. Furthermore, the previous misprint for the factor in front of the double integral in Ref. 26 is shown in the Appendix.

\[ \bar{\psi}_{th} = \frac{\psi_{th}}{\psi_0} = \left( 1 + \frac{c_{b1} - u_1}{u_i} \right) \bar{s}_t, \]

(2)

and the initial dimensionless ripple amplitude at the rarefaction head is

\[ \frac{\psi_{th}}{\psi_0} = 1 + \frac{c_{b1} - u_1}{u_i}, \]

(3)

where we see that the initial amplitude at the head does not depend on the properties of fluid \( a \), but the initial amplitude of the ripple at the tail depends on the thermodynamic properties of both fluids. After some laborious algebra, the linearized equations of motion can be solved to give us the tangential velocity there, which in turn modifies the tangential velocity profile inside the rarefaction as a function of the coordinates \( \xi \) and \( \eta \). The details are omitted as they can be found elsewhere:

\[ \bar{\partial}_v(\xi, \eta) = \sqrt{\xi} \int_1^\xi J_0[\sqrt{n} \eta(\xi - z)] \partial v_1(z) dz, \]

(4)

where the number \( n = (\gamma_b + 1)/(3 - \gamma_b) \). The function \( \partial v_1 \) is related to the derivative of the initial tangential velocity profile \( \partial v_1(\xi) = d\partial v_0(\xi)/d\xi \), and where \( \partial v_0(\xi) = \partial v[\bar{\xi}], t = 0 + \) \( / \sqrt{\xi} \). It is not difficult to see that \( \partial v_1(\xi) = \sum_{j=0}^{\infty} \alpha_j \xi^j \), where the coefficients \( \alpha_j \) and the exponents \( \epsilon_j \) are shown in the Appendix.

### B. Evolution equation for \( \bar{\psi}_{tr}(\tau) \)

Once the rarefaction has been created, the leading and trailing fronts move with the imposed corrugations. For the case of the rarefaction head, as no sound waves arrive from behind, the value of its initial ripple can not be changed and therefore it remains constant in time.\(^{20-23}\) On the contrary, the sound perturbations generated at each slice of fluid inside the rarefaction travel downwards and reach the trailing front, where they will finally be transmitted toward the contact surface. When arriving to the trailing edge, the sound fluctuations modify the tangential velocity there, which in turn modifies the amplitude of the ripple itself. With the theory developed in Ref. 26, the exact solution for the tail ripple as a function of time can be rewritten as

\[ \bar{\psi}_{tr} = C_1 + \frac{\gamma_b - 3}{4M_1} \bar{\partial}_v(z) + \xi_{st} \sqrt{\eta(\xi - z) / 4} \bar{\partial}_v(z) \tau B(\tau), \]

(5)

where the function \( B(\tau) \) is defined as

\[ B(\tau) = \int_1^{\xi_{st}} dz \frac{\partial v(z)}{\sqrt{\xi_{st} - z}} \int_0^{\xi_{st}} \left( J_0 \bar{\psi}_{tr}(\tau) + \xi_{st} \sqrt{\eta(\xi_{st} - z) / 4} \right) B(\tau) \]

(6)

A previous misprint for the factor in front of the double integral in Ref. 26 has now been corrected. Furthermore, the constant \( C_1 \) is given by \( C_1 = \bar{\psi}_{tr} + (3 - \gamma_b)/(4M_1) \tau \). The function \( \bar{\partial}_v(0) \) is the tangential velocity fluctuation at the trailing edge, and is given by Eq. (4). The function \( \bar{\partial}_v(0) \) is the same function but at \( \tau = 0 + \), and is given by Eq. (1).

In Fig. 2 we show the ripple amplitude as a function of time for a case in which \( \gamma_b = \gamma_0 = 5/3 \), the preshock density ratio is \( R_0 = 1/100 \), and a very strong incident shock with Mach number \( M_1 \gg 1 \) has collided with the interface. We see that after some gentle oscillations, the ripple amplitude reaches a constant rate of growth. The minimum value of the rarefaction Mach numbers is \( M_1 \approx 0.5350 \). At first, we note that the late time growth is linear, as already known from previous works.\(^{20-22,23,26}\) Besides, we recognize that the line that gives the asymptotic growth passes through the origin, when extrapolated backwards in time. This fact will be discussed in the next section as it is an important point to decide the asymptotic behavior of the tail ripple in freeze-out.

### C. Late time growth of the trailing edge ripple

As can be easily checked, the function \( B(\tau) \), defined in Eq. (6) reaches a constant value, for \( \tau \gg 1 \). This means that the ripple amplitude reaches a constant velocity of growth for large times. This result can also be deduced qualitatively, by realizing that the zero-order Riemann invariant \( J_0(\xi) \), when evaluated at the deformed trailing front \( (c_{b1} + \psi_{tr}) \), tends to a constant, as discussed in Ref. 26. This is only possible if \( \psi_{tr} \propto \tau \) because of the self-similarity of the background flow. The value of that velocity can be easily deduced by differentiating Eq. (5) in the limit \( \tau \rightarrow \infty \) to get

\[ \frac{d\bar{\psi}_{tr}^{\infty}}{d\tau} = \frac{\xi_{st} \sqrt{\eta(\xi_{st} - z) / 4}}{\sqrt{\xi_{st} - z}}. \]

(7)

The integral can always be evaluated analytically by means of known functions (hypergeometric functions) and this has been extensively discussed in Ref. 26. In this section we want to add some additional results that are apparent from Fig. 2 and also in Figs. 4 and 7 of Ref. 20, or Fig. 15 of Ref. 22. It is easy to see that the straight line defining the ripple...
amplitude for large times, passes through the origin, when extrapolated to \( \tau \to 0^+ \). That is, for large times, the tail ripple behaves like \( \psi_{\text{r}} = \chi_\text{r} \tau \). However, looking at Eq. (5) and taking the limit \( \tau \to \infty \), we see that the growth seems to be of the form \( \psi_{\text{r}} = \sigma_1 + f(\tau) \), where \( f(\tau) \approx 1 \), and \( \sigma_1 = \bar{\psi}_\text{r} + (3 - \gamma_b) / (4M_1) \bar{\psi}_{\text{r}} = 0 \). This last result contradicts our previous observation of zero ordinate to the origin. Actually, as will be shown later on, it is the last integral in Eq. (5) that is adding an additional constant for large times, which cancels out the quantity \( \sigma_1 \).

1. Asymptotic scaling of \( dB/d\tau \)

It is better to rearrange terms in the definition of \( B \) and perform the integral on \( z \) at first. Its derivative \( (dB/d\tau) \) can be easily deduced:

\[
\frac{dB}{d\tau} = M_1^{\alpha^2/2} \int_1^{\xi_\text{rt}} dz \ \
\times \frac{J_n[M_n^2(\xi_\text{rt} - z)]}{t_n^{1/n} M_n^2(\xi_\text{rt} - z)}. \tag{8}
\]

To integrate inside the rarefaction fan, we recall the integral representation of the Bessel function in the complex plane:

\[
\frac{J_n(x)}{x^n} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{-\omega x} x^n \frac{d\omega}{\omega^{n+1}}, \tag{9}
\]

where \( \nu \) is an arbitrary real or complex number, and the contour of integration is a curve that encircles the origin once, anticlockwise. Using this result into Eq. (8), we get

\[
\frac{dB}{d\tau} = \sum_{j=0}^{2} \alpha_j \bar{\psi}_j M_1^{\alpha^2/2} \int_1^{\xi_\text{rt}} dz \ e^{-2 \omega_{\text{b}} M_1^2 K_j(\omega) d\omega}, \tag{10}
\]

where \( \phi(\omega) = n \omega^2 / (4\pi) \), and the factor \( K_j \) is defined as

\[
K_j(\omega) = \int_{1}^{\xi_\text{rt}} e^{-\omega_{\text{b}} M_1^2 z} dz. \tag{11}
\]

The terms \( K_j \) can be evaluated by means of confluent hypergeometric functions. The details are omitted here. The temporal asymptotic limit can then be easily taken and the late time behavior of \( B(\tau) \) be found. After some algebra, we find

\[
\frac{dB}{d\tau} = \frac{\sigma_2}{\tau} - \frac{\delta B_j(1)}{2M_1^{\alpha^2/2} n} J_{n}^{1/2}(nM_1^2 - M_1^2) \tag{12}
\]

where \( \sigma_2 = \delta \omega_{\text{r}}(\xi_\text{rt}) / (2M_1^{\alpha^2/2} n) \). We see that \( B(\tau) \) has the desired asymptotic behavior. Going back to the expression for \( \bar{\psi}_{\text{r}} \) in Eq. (5), taking the asymptotic limit (\( \tau \to 0^+ \)), using the results of this paragraph and rearranging terms, we obtain

\[
\bar{\psi}_{\text{r}}(\tau) = \left[ \bar{\psi}_{\text{r}} + \frac{3 - \gamma_b}{4M_1} \delta \omega_{\text{r}} + (2\gamma_b - 3/4) \bar{\psi}_{\text{r}} \delta \omega_{\text{r}} \right] + \chi_\text{r} \tau + F(\tau), \tag{13}
\]

where \( F(\tau) \) is an oscillating function of time whose amplitude decays like \( 1/\sqrt{\tau} \). The ripple velocity then behaves asymptotically like a constant plus an oscillatory term that decays like \( 1/(\tau \sqrt{\tau}) \). However, the important point is to note, after some algebra, that the first term inside brackets in Eq. (13) is exactly equal to zero for any initial conditions (fluids compressibilities, incident shock strength, or preshock density jump). Therefore, the trailing edge ripple scales asymptotically as

\[
\bar{\psi}_{\text{r}} = \chi_\text{r} \tau, \tag{14}
\]

which means that the ordinate to the origin is equal to zero, in any situation. This finding confirms the behavior realized in Fig. 2 for a particular set of preshock conditions. Furthermore, the actual form of the function \( \delta \omega_{\text{r}}(\xi) \) inside the rarefaction does not matter at all, for the scaling indicated in Eq. (14) to hold. This means that this asymptotic behavior is common to all corrugated rarefactions moving into undisturbed fluids, whether they have been generated after an incident shock refracted at the interface, or whether they were created by the sudden release of an initial pressure difference. An important additional conclusion inferred from Eq. (14) is the fact that in freeze-out, the tail ripple reaches zero amplitude in any situation. This means that freezing-out trailing rarefaction fronts are superstable in the same way as corrugated shocks propagating into undisturbed ideal gases.

D. Asymptotic rate of growth of the tail ripple

As has been found in the last paragraph, the tail ripple grows linearly with zero ordinate for large times. According to Eq. (7), in the general case, the asymptotic growth rate can be rewritten as

\[
\chi_{\text{rt}} = \xi_{\text{rt}} \gamma_b - 3 \sum_{j=0}^{2} \alpha_j e_i I_j, \tag{15}
\]

where the integral \( I_j \) is given by

\[
I_j = \int_{1}^{\xi_{\text{rt}}} e^{-j/\xi_{\text{rt}}} dz. \tag{16}
\]

For the details of the evaluation of the integrals \( I_j \), the reader is referred to Ref. 26. They can be expressed in terms of hypergeometric functions. The cases in which either \( e_1 \) or \( e_2 \) are negative integers require a separate study. In any case, the integrals can always be calculated analytically. In Fig. 3 we show the asymptotic value of the dimensionless rate of growth, as calculated from Eq. (5) for a situation in which \( \gamma_b = 5/3 \) and \( R_0 = 1/100, 1/1000, 1/10000 \), respectively. The horizontal axis is the rarefaction strength \( M_1 \). The rarefaction Mach number is varied by changing the incident shock intensity, as discussed in Sec. I. At infinite intensity we reach the minimum possible value \( M_1^\text{m} \) which, for the cases considered in Fig. 3 are given by \( M_1^\text{m} = 0.5350 \) and \( 0.3583 \), respectively. The corresponding points are indicated with a letter P in the curves shown in Fig. 3. For any other combination of ideal gases at the contact surface, the procedure is essentially the same. There is, however, an interesting property for the rarefactions generated as shown in Fig. 1. We see that the ripple growth is always positive for the case \( R_0^\text{m} = 1/100 \), but it changes sign for \( R_0^\text{m} = 1/10000 \). There should be an intermediate value for the preshock density ratio, and finite values for the other parameters \( (\xi_{\text{r}}, \gamma_{\text{r}}, \gamma_b) \) at which the rate
of growth vanishes asymptotically (that is, $\chi_0 = 0$). We call this situation freeze-out of the tail ripple growth. The first person to use this name has been Mikaelian, who coined it to indicate a similar phenomenon occurring at the contact surface.\(^{27}\) In this last case, we speak of freeze-out of the RMI. The reasons for the freezing-out of the RMI at the contact interface should be sought in the interaction between the rarefaction fan, the transmitted shock, and the material surface. In the case that concerns us here, we are only interested in the freezing-out of the perturbations at the tail ripple. This surface is not influenced by the growth at the material contact surface, and therefore, the physical reasons for freeze-out at the trailing edge should be intrinsic to the growth of the perturbations inside the rarefaction fan. This does not preclude the possibility that freeze-out at the contact surface is somehow influenced by freeze-out at the rarefaction. In fact, this could be certainly the situation for cases in which fluid $a$ has very low density, as has been recently reported.\(^5\) Freeze-out at the contact surface is certainly a very important subject of research, which has not been studied in any detail yet for a situation like that depicted in Fig. 1. Its analysis is beyond the objectives of the present work and is left for future investigations. We content ourselves, for the moment, in studying the freeze-out at the rarefaction tail only. As will be unambiguously shown later, the way in which the tangential velocity perturbations at $t=0^+$ are distributed between the tail and the head is the key factor that decides whether or not freeze-out at the rarefaction tail will be achieved for large times. For the problem of interest in this work, let us see, very qualitatively, why and when should there be freeze-out at all, by looking closer to the function $\chi_0(M_1)$ in both interesting physical limits: weak and strong rarefactions. As we know from a previous work,\(^{20}\) in the limit $M_1 \to 1$, the growth rate scales like $\sqrt{1-M_1}$.\(^{20}\) Furthermore, as has been recently shown, the approximate scaling in that limit takes the form\(^{26}\)

$$\chi_0 \propto \sum_{j=0}^{2} \alpha_j P_j = 0. \quad \text{(19)}$$

By looking at the expression given by Eq. (19), it is clear that it is quite difficult to get the preshock quantities that lead to freeze-out in a simple form. The best strategy is to formulate an iteration procedure which gets closer to the freezing-out conditions at each iteration step. It will be seen that, given a pair of isentropic exponents ($\gamma_{a,b}$) we should determine the maximum rarefaction intensity (corresponding to a shock of infinite intensity), at which there is freeze-out. This defines a maximum preshock density jump ($R_0^{\text{max}}$), which is a

\[ (\chi_0^n)_{M_1 < 1}^{\gamma} = \frac{1}{2} \sum_{j=0}^{2} \left( \frac{\gamma_b - 3}{\Gamma(e_1 + 2)} M_1 \right) \]

\[ \times \left( \frac{\gamma_b - 3}{\Gamma(e_1 + 2)} M_1 \right) \]

That is, we have $\chi_0 \approx \theta(1-M_1)$, where $\theta = \theta(\gamma_b, \gamma_a, R_0, z_i)$. To evaluate it, we consider a shock of negligible intensity ($z_i \to 0$). In this limit, $\theta$ only depends on $\gamma_b$ and can be seen to be always a positive quantity. This means that the growth near the limit of very weak rarefactions is always positive, irrespective of the values of the fluids parameters.

Let us now consider the growth in the limit of a very strong rarefaction, when $M_1 \to 0$. From the asymptotic scaling deduced in Ref. 26, the growth can be approximated written as

$$\chi_0^{\gamma} \approx \alpha_1 e_1 \gamma_b - 3 \sqrt{\pi} \Gamma(e_1) M_1$$

\[ + \alpha_2 e_2 \gamma_b - 3 \sqrt{\pi} \Gamma(e_2) M_1^{2} \]

\[ + \gamma_b - 3 \sqrt{\pi} \Gamma(e_1) M_1^{2} \]

\[ \times M_1^{3-\gamma_b/(2\gamma_b-1)} + \frac{\gamma_b - 3}{4} \sqrt{\pi} \gamma_0 - \frac{3 \alpha_1 e_1}{8 (e_1 + 2)} \]

\[ - \frac{3 \alpha_2 e_2}{8 (e_2 + 2)} \]

\[ M_1^{3-\gamma_b/(2\gamma_b-1)} + \ldots \quad \text{(18)} \]

For any given $R_0 \ll 1$, the limit $M_1 \to 0$ will be approached for a shock with infinite intensity. Then, we can approximate the values of the coefficients $\alpha_j$ (shown in the Appendix) by considering the limit of a very strong incident shock. The resulting expression is only dependent on $\gamma_b$ and $M_1$. We have kept the first nonvanishing terms in the asymptotic expansion near $M_1 \sim 0$. It can also be easily checked that the growth near $M_1 \to 0$ is always negative below some critical $\gamma_b \approx 2.2913\ldots$. Thus, the growth rate has to change sign at some intermediate rarefaction strength.

III. FREEZE-OUT AT THE TRAILING EDGE

In the previous sections we have developed the basic steps that allow us to search for the zones in the space of initial parameters that lead to asymptotic freeze-out:

$$\chi_0 \propto \sum_{j=0}^{2} \alpha_j P_j = 0. \quad \text{(19)}$$

\[ (\chi_0^n)_{M_1 < 1}^{\gamma} = \frac{1}{2} \sum_{j=0}^{2} \left( \frac{\gamma_b - 3}{\Gamma(e_1 + 2)} M_1 \right) \]

\[ \times (\gamma_b - 3 \sqrt{\pi} \Gamma(e_1 + 2) M_1^{2}) \]

\[ \gamma_b - 3 \sqrt{\pi} \Gamma(e_1 + 2) M_1^{2} \]

\[ \times M_1^{3-\gamma_b/(2\gamma_b-1)} + \frac{\gamma_b - 3}{4} \sqrt{\pi} \gamma_0 - \frac{3 \alpha_1 e_1}{8 (e_1 + 2)} \]

\[ - \frac{3 \alpha_2 e_2}{8 (e_2 + 2)} \]

\[ M_1^{3-\gamma_b/(2\gamma_b-1)} + \ldots \quad \text{(18)} \]

\[ \times M_1^{3-\gamma_b/(2\gamma_b-1)} + \ldots \quad \text{(18)} \]

\[ \times \left( \gamma_b - 3 \sqrt{\pi} \Gamma(e_1 + 2) M_1^{2} \right) \]

\[ \gamma_b - 3 \sqrt{\pi} \Gamma(e_1 + 2) M_1^{2} \]

\[ \times M_1^{3-\gamma_b/(2\gamma_b-1)} + \frac{\gamma_b - 3}{4} \sqrt{\pi} \gamma_0 - \frac{3 \alpha_1 e_1}{8 (e_1 + 2)} \]

\[ - \frac{3 \alpha_2 e_2}{8 (e_2 + 2)} \]

\[ M_1^{3-\gamma_b/(2\gamma_b-1)} + \ldots \quad \text{(18)} \]
function of both isentropic exponents. Freeze-out would then be guaranteed for \( R_0 < R_{0,\text{max}} \) if the corresponding shock Mach number (\( M_s \)) is found by iteration from Eq. (19). Besides, it will be shown that freeze-out can only be found for \( \gamma_b < \gamma_{cr} = 2.2913 \ldots \)

### A. Iteration

The space of initial parameters, being four dimensional, is too large to start blindly searching for freeze-out conditions. From knowledge accumulated in previous works, we should be able to focus our search, and narrow the zones to be explored. In fact, for a given \( \gamma_b < 3 \), there are still many possible combinations of the other three parameters (\( \gamma_s, R_0 \), and \( z_t \)). Paying attention to Fig. 3, we realize that for a given value of the density jump (for example, \( R_0 = 1/10 \)), the growth is positive all the way from weak incident shocks \( (z_t = 0, M_1 = 1) \) up to very strong shocks \( (z_t = \infty, M_1 = M_{11}) \).

Let us call \( P \) the point that corresponds to the growth when \( M_1 = M_{11} \). This point \( P \) is the ending point in the ripple velocity curve, that starts from zero at \( M_1 = 1 \). If we reduce \( R_0 \) further (for example, \( R_0 = 1/10,000 \)), the growth will be negative. The point \( P \) has moved to a new position in the \( (M_1, \chi_{\text{cr}}) \) plane, below the horizontal axis, and the asymptotic growth becomes negative. Then, a good criterion with which to start the search, for a prescribed \( \gamma_b < 3 \), is to ask for the conditions that make the point \( P \) lie exactly at the horizontal axis, that is, \( \chi_{\text{cr}}(P) = 0 \). This point would correspond to a shock of infinite intensity and then, the analytical formula for the growth rate at \( P \) would only depend on \( \gamma_b \) and \( M_1 \). In fact, by taking the limit \( z_t \to \infty \) we can deduce that the growth rate in Eq. (7) does only depend on \( \gamma_b \) and \( M_1 \), but not on \( R_0 \) and \( \gamma_{cr} \).

In Fig. 4 we show the behavior of \( \chi_{\text{cr}}(P) \) as a function of \( M_1 \) for different values of \( \gamma_b \). All of these curves, are in some sense, kind of limiting growth curves, for a given \( \gamma_b \). Besides, we see that for \( \gamma_b \) above some critical value \( \gamma_{cr} \) there is no possibility of freeze-out, as \( \chi_{\text{cr}}(P) > 0 \). We have found \( \gamma_{cr} = 2.2913 \ldots \). We also see that for \( \gamma_b \) near \( \gamma_{cr} \) there is the possibility of two freeze-out points. The second freeze-out occurs for very low rarefaction intensities: \( M_1 \leq 0.02 \) and \( \gamma_b \) very near \( \gamma_{cr} \). Summing up, for \( \gamma_b < \gamma_{cr} \) there is always a rarefaction strength (which corresponds to an incident shock with infinite Mach number), at which we will find freeze-out. According to Eq. (15), this is equivalent to require the existence of \( M_{11} \) such that

\[
\sum_{j=0}^{2} \alpha_j \gamma_b^{M_{11}, z_t} 1) \epsilon_j (\gamma_b) l_j (\gamma_b, M_{11}) = 0, \tag{20}
\]

where use has been made of Eq. (15), and where the explicit dependence on \( \gamma_b \) and \( M_{11} \) has been stressed, as well as the fact that calculation is being done in the limit \( z_t \to 1 \). It is not possible to solve the above equation analytically for any value of \( \gamma_b \). Thus, the only possibility is to start an iterative procedure. With mathematical software, this can be easily implemented. In Fig. 5 we show the function \( M_{11} = M_{11}(\gamma_b) \), for those values of \( \gamma_b \) which exhibit only one freezing-out point. For a given value of \( \gamma_b < \gamma_{cr} \), we determine \( M_{11} \) according to Eq. (20). For that value \( M_{11} \), we can calculate, for any given \( \gamma_{cr} \), the corresponding \( R_{0,\text{max}} \) which makes \( \chi_{\text{cr}}(P, \gamma_{cr}, \gamma_b, R_{0,\text{max}}) = 0 \). Then, according to the discussion of the previous sections, freeze-out will only be possible for \( R_0 < R_{0,\text{max}} \). At any value \( R_0 \) lower than \( R_{0,\text{max}} \), the incident shock intensity is now finite. The question is: how much can we decrease the incident shock intensity and still find freeze-out?

With the results obtained in Ref. 26, it is not difficult to see that by decreasing the transmitted shock intensity \( z_t = (p_T - p_0)/p_0 \), where \( p_0 \) is the initial pressure and \( p_T \) is the pressure behind the transmitted shock front), and \( R_0 \) simultaneously to zero, while keeping the incident shock intensity fixed, we can reach a minimum incident shock Mach number for which there is freeze-out. In this other limit, the rarefaction perturbation profiles do also only depend on \( \gamma_b \) and \( M_1 \).

This defines another limiting curve for the ripple growth \( \chi_{\text{cr}} \).

Along this curve, we have \( z_t \to 0 \). It will cross the horizontal axis at another point \( M_1 = M_{10} \). Therefore, given \( \gamma_b \) we would only find freeze-out for rarefaction intensities that satisfy \( M_{10} < M_1 < M_{11} \). In Fig. 6 we plot the asymptotic growth rate in the mentioned limit \( (z_t \to 0) \), as a function of \( M_1 \) for...
different values of $\gamma_b$. Given $\gamma_b$ we can determine the lower limit $M_{10}$, by requiring the vanishing of $\chi_0$ in the limit $z_t \to 0$, which is equivalent to ask for

$$\sum_{j=0}^{2} \alpha_{j}(\gamma_{b}, M_{10}, z_t) \approx 1) \epsilon_{j}(\gamma_{b}, M_{10}, M_{10}) = 0, \quad (21)$$

where the explicit dependence on $\gamma_{b}$ and $M_{1}$ has been emphasized. In Fig. 7, we show the behavior of $M_{10}$ as a function of $\gamma_{b}$. We have chosen the interval $1.45 \leq \gamma_{b} \leq 2.1$ as a good representative one, as experiments on the RMI (rarefaction case) in the recent past have chosen, for example, $\gamma_{b} \sim 1.45$ and 1.8. Thus, the task of focusing into the regions of freeze-out, consists at first, of determining the limiting rarefaction strengths, for a given value of $\gamma_{b} < \gamma_{c}$. For a given value of $\gamma_{b}$ and with the calculated value of $M_{11}$, we can determine the maximum density ratio $R_{0}^{\text{max}}$. From this point, we reduce the density ratio up to the limit $R_{0}^{\text{max}}$ to reach the minimum rarefaction strength $M_{10} \neq 0$. This minimum value for $M_{11}$ defines a minimum possible incident shock Mach number $M_{10}^{\text{min}}$, below which it will not be possible to find freeze-out, except the trivial freezing-out point $M_{10} = 0$. It is also noted that $M_{10}^{\text{min}}$ only depends on $\gamma_{b}$. In Fig. 8, we show the quantity $M_{10}^{\text{min}}$ as a function of $\gamma_{b}$. The next task is, given a pair of values for $\gamma_{a}$ and $\gamma_{b}$, to plot the freeze-out contours $M_{1}=M_{1}(R_{0})$. These plots are defined in the intervals: $0 < R_{0} < R_{0}^{\text{max}}$, $M_{10}^{\text{min}} < M_{1} < \infty$. Both intervals are equivalent to a corresponding variation for the rarefaction intensity parameter: $M_{10} < M_{1} < M_{11}$. To get a freeze-out point $(R_{0}, M_{1})$ we must use the condition for freeze-out, that is, $\chi_{0} = 0$, from Eq. (7). This last equation is evaluated at $M_{1}$, satisfying $M_{10} < M_{1} < M_{11}$. It is clear that an analytic solution would be impossible in most of the cases of interest. Therefore, the only alternative is to solve them by iteration. Roughly speaking we start with a value for $R_{0}$ a bit smaller than $R_{0}^{\text{max}}$ and start iterating, searching for the incident shock Mach number that keeps the asymptotic growth rate below a small enough level. It is seen that, if we keep $R_{0} < 10^{-10}$, the incident shock Mach number can be calculated with more than four accuracy digits. In Fig. 9(a) we show the freeze-out contour plots: $M_{1}=M_{1}(R_{0})$ for $\gamma_{b}=3/2$ and three different values of $\gamma_{a}$: 6/5, 3/2, and 5/3. In Fig. 9(b) we show the contour plots: $M_{1}=M_{1}(R_{0})$ for the same combination of gases. The freeze-out, for a given density jump and given pair of $\gamma$ values, is only possible for sufficiently strong shocks (besides the trivial freezing-out conditions exist at $M_{1}=0$ and 1). Furthermore, the minimum shock intensity below which there is no freeze-out (no matter how small the density ratio or the values of the isentropic exponent), does only depend on the compressibility of the expanding fluid. However, the maximum density ratio, above which there can be no freezing-out, is dependent on the compressibilities of both fluids.

**B. Temporal evolution at freeze-out**

In Fig. 10 we show the temporal evolution of a freeze-out case, with $\gamma_{a} = 5/3, R_{0} = 0.004987$, and $M_{1} \gg 1$. We see that the tail ripple executes decaying oscillations which seem to vanish for large times. This is in agreement with the conclusion of Sec. II. In fact, we know that there is no asymptotic ordinate to the origin for the function $\psi_{b}(\tau \gg 1)$ for any initial condition, according to the results embodied in
Eq. (14). In freeze-out, the asymptotic rate of growth is zero, and it is therefore clear that the asymptotic ripple amplitude is also zero. We could express this result by saying that the freezing-out rarefaction trailing front is superstable, as much as a corrugated shock, when moving into an undisturbed ideal gas.\textsuperscript{1,2,5,18,21}

### C. Qualitative discussion

We have at our disposition an exact analytical model with which to study the perturbation growth in linear theory. Therefore, it is expected that it sheds some light regarding the mechanisms that drive the instability growth toward freeze-out.\textsuperscript{20,26} We would like to obtain a qualitative criterion that indicated to us whether a given combination of preshock parameters would possibly result in freeze-out or not. At first, we note here that the asymptotic rate of growth can also be expressed as a time integral of the tail tangential velocity. Indeed, it is easy to show that

\begin{equation}
\chi_t^\infty = M_1 \left( \frac{\gamma_b + 1}{4} \right) \int_0^\infty \delta \omega_l(t) dt. \tag{22}
\end{equation}

Thus, the rate of growth of the ripple at the trailing front is the result of the cumulative sum of all the velocity perturbations that arrived to the rarefaction tail during instability evolution. The important point to be understood from Eq. (22) is that the value of the asymptotic rate of growth does depend on the details of what happened inside the rarefaction fan since early times. In other words, as also stressed by Velikovich and Phillips, initial conditions are very important in order to get the correct asymptotic growth rate of the perturbations.\textsuperscript{20} In the case of the rarefaction here, the tail only interacts with the upstream layers of fluid, receiving all the perturbations that come from the slices with $A > M_1$. In freeze-out, $\chi_t^\infty$ is zero, because the integral in the right-hand side of Eq. (22) is zero. However, for a qualitative discussion we have found it better to examine the expression for the ripple velocity as given in Eq. (7). It is instructive to study the integral on the right hand side of that equation, before restructuring it as a sum of three separate integrals, as has been later done in Eq. (15). We can recognize, that if the total integral in Eq. (7) has to be zero, it is because its integrand changes sign at least one time in the integration interval. In this case, it is $\delta \omega_l(z)$ that must change sign in the range $1 \leq z \leq \zeta_{0v}$. And the corresponding positive and negative areas should be of equal magnitude. Physically, this means that the initial tangential velocity profile, as given by Eq. (1), has an extremum, which could be a local maximum or minimum. To see that this is the case, we plot the initial tangential velocity profile $\delta \omega_l(\zeta, t=0+)$ in the interval $M_1 \leq \zeta \leq \zeta_{0v}$ in Fig. 11. We see that there is a local minimum between both fronts. The minimum is located at such a position inside the rarefaction fan, that there is an exact cancellation of the positive and negative areas either in Eq. (7) or, equivalently, in the temporal evolution integral given by Eq. (22). Then, the existence of such a minimum is a necessary condition to decide whether or not the late time evolution is one of freeze-out. Rarefactions generated in such a way that the initial tangential velocity profile does not show an extremum
can never result in asymptotic freeze-out. Of course, the existence of that minimum is not enough for the asymptotic vanishing of the tail ripple at all. A case in which there is no such minimum inside the rarefaction fan, is the rarefaction generated by the sudden release of a diaphragm separating two gases at rest with different initial pressures. This situation has been discussed by Velikovich and Phillips. In fact, for the diaphragm case, the initial tangential velocity profile has the simpler form $\delta v(\xi, t=0+) = 2(\xi - \xi_0)/(\gamma_b + 1)$, which is a straight line in terms of the $\xi$ coordinate and does not show any extremum inside the interval $M_1 < \xi < \xi_0$. Thus, the possibility of freeze-out is excluded for any combination of the initial gas parameters.

IV. SUMMARY

An analytical model to study the freeze-out at the rarefaction tail has been presented. It has been shown that for a rarefaction generated with an initial shock that crosses a contact surface, there is freeze-out for $\gamma_b < \gamma_c \sim 2.2913$. To see freeze-out for any $\gamma$ below this critical value, the rarefaction intensity should be bounded between a minimum and a maximum strength: $M_{10}(\gamma_0) < M_1 < M_{11}(\gamma_0)$. The rarefaction intensity $M_{11}$ defines, after specifying the value of $\gamma_a$, the value of $R_{0\max}$. The quantity $M_{10}$ defines the minimum value of the shock intensity for which there is freeze-out $M_{\min}$. Then, the subset of the space of initial parameters in which it is possible to get freeze-out conditions is defined by $0 < R_0 < R_{0\max}$, $M_{\min} < M_1 < \infty$, and $\gamma_b < \gamma_a \sim 2.2913$. It is also seen that the rarefaction tail ripple decays to zero amplitude asymptotically in time, for freezing-out conditions. Besides, it has been shown that a necessary condition for asymptotic freeze-out of the tail ripples is the existence of a maximum or a minimum of the initial tangential velocity profile inside the rarefaction zone.

ACKNOWLEDGMENTS

Encouragement from A. R. Piriz is gratefully acknowledged.

This work has been partially supported by Ministerio de Educacion y Ciencia, Spain, FTN2003-00721.

APPENDIX: COEFFICIENTS $\alpha_j$ AND $\epsilon_j$

The coefficients $\alpha_j$ are

$$
\alpha_0 = -\frac{\xi_0^2}{\gamma_b + 1} + \frac{\xi_0}{\gamma_b + 1} \left( \frac{c_{bl}}{M_1 - \vartheta_b} \right)
+ \frac{4M_1}{\gamma_b - 1} \left( \frac{c_{bl}}{\sqrt{\vartheta_b}} \right) - \frac{4M_1^2}{\gamma_b - 1} \frac{c_{bl}}{\sqrt{\vartheta_b}},
$$

(A1)

$$
\alpha_1 = \frac{2}{\gamma_b - 1} \left( \frac{c_{bl}}{M_1 + \vartheta_b^{1/2}} \right) - \frac{4M_1^2}{\gamma_b - 1} \frac{c_{bl}}{\sqrt{\vartheta_b}},
$$

(A2)

$$
\alpha_2 = \frac{\gamma_b + 1}{\gamma_b - 1} \frac{c_{bl}}{\sqrt{\vartheta_b}},
$$

(A3)

and the exponents $\epsilon_j$ are given by $\epsilon_0 = -1/2$, $\epsilon_1 = 1/2 (\gamma_b + 1)/(\gamma_b - 3)$, and $\epsilon_2 = 1/2 (\gamma_b - 1)/(\gamma_b - 3)$.

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