Beam plasma electromagnetic instabilities in a smooth density gradient: Application to the fast ignition scenario

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The integrated growth rate of various relativistic beam/plasma instabilities in a weakly varying plasma density gradient is calculated using a WKB-like approximation. It is proven that such an assumption can be made in fast ignition scenario conditions. The formalism is applied to the two-stream, the filamentation, and the two-stream/filamentation instabilities, the latter instability being a mixture of the former two, and is the fastest growing one. The results are restricted to collisionless plasmas and nonrelativistic beam and plasma temperatures. Filamentation instability is reduced by the density gradient and eventually does not develop in the core, whereas two-stream and two-stream/filamentation instabilities should saturate even before they feel the gradient. Various effects connected to the density gradient are discussed. It is found they should be negligible as long as these later instabilities remain in their respective linear regime. © 2005 American Institute of Physics. [DOI: 10.1063/1.2084907]

I. INTRODUCTION

Beam plasma instabilities are a central issue in plasma physics and many investigation efforts have been produced recently in connection with the fast ignition scenario (FIS).\textsuperscript{1} This scenario gives a very important role to the relativistic electron beam (REB) generated by an ultrashort and intense laser pulse. The REB travels from the outer region of the precompressed pellet to the deuterium-tritium (DT) core where it is supposed to ignite the fuel.\textsuperscript{2} While traveling through the plasma surrounding the DT core, the beam quickly prompts a return current\textsuperscript{3} so that the resulting configuration is that of two streams passing through a plasma. This very classical system suffers from some well-known instabilities such as the two-stream and the filamentation instabilities, and some not so well known instabilities, such as the ones which are found for wave vectors arbitrarily oriented.\textsuperscript{4–6} Many numerical, theoretical, and experimental efforts have been devoted to the investigation of these instabilities recently,\textsuperscript{7–12} but as far as theoretical works are concerned, almost all of them are restricted to plasmas without density gradients. This is due to the extreme complexity of analytical calculations generated when trying to compute an electromagnetic dispersion equation with a varying plasma density, even in the linear regime. Our objective is to derive a simple formula for the integrated growth rate of a given unstable mode when the density gradient is small. Considering the scale lengths set by the main unstable modes of the system, we show that they all remain much smaller than the characteristic variation length of the density gradient in the FIS. This allows the use of a WKB-like approximation to deal with the growth rate along the gradient. Some limitations of this method are finally presented in the Conclusion. The growth rate expressions we use in the sequel have been derived in the limit of small (nonrelativistic) beam and plasma temperatures.\textsuperscript{13} Although they are valid for a relativistic beam, the reader should keep in mind that higher temperatures, and especially higher beam temperature, should have an important effect upon the present results.

II. GROWTH RATE AND DENSITY GRADIENT

Let us consider an unmagnetized plasma at temperature \( T_p \) where the electronic density \( n_e \) varies along the \( z \) axis (see Fig. 1). Ions are supposed to form a fixed neutralizing background. We then suppose that a beam of density \( n_b \), temperature \( T_b \), and relativistic velocity \( V_b \| z \) is passing through the plasma so that it is traveling along the density gradient. The system is considered as infinite and homogenous in the direction normal to \( V_b \), and we shall give the condition required for such a statement. We use the typical relativistic notations \( \gamma_b = 1/\sqrt{1-\beta^2} \) with \( \beta = V_b/c \), and introduce some dimensionless quantities that we shall use throughout the paper,

\[
\alpha = \frac{n_b}{n_e}, \quad \rho_b = \frac{V_{th}}{V_b}, \quad \rho_p = \frac{V_p}{V_b},
\]

where \( V_{th} \) and \( V_p \) are the plasma and beam thermal velocities, respectively. So far, these dimensionless quantities are also functions of \( z \).

We now turn to the unstable mode, the development of which through the density gradient is studied. It is important to remark that although there is a whole continuum of unstable modes, it makes perfect sense to speak of only one of...
them, namely, the fastest growing, because this is the one that will shape the beam while the others make corrections to this basic shape. As a matter of fact, experiments or simulations of beam plasma interaction can be perfectly interpreted in terms of the growth of one mode, although others may have grown. If we now consider the most unstable mode, it is characterized by its growth rate $\delta$ and its wave vector $k$, the coordinates of which with respect to the beam are $(k_\perp, k_\parallel)$. The growth rate and the wave vector are obviously functions of $z$ because they are usually functions of the beam and plasma densities and temperatures (see the Conclusion for a discussion of this issue). The typical distances to be considered if we are to use a WKB-like approximation are the inverse of the wave-vector coordinates on one hand $L_{k_\parallel} = 1/k_\parallel$ and $L_{k_\perp} = 1/k_\perp$, and on the other hand $\lambda$, the typical scale of density variation along $z$ with $1/\lambda = (\partial n_e/\partial z)/n_e$. The system is considered as infinite in the normal direction so that there is no condition related to $L_\perp$. As for the parallel direction, in which the density varies, the problem can be considered as locally infinite if $L_{k_\parallel} \ll \lambda$. This means that if the wave-vector characteristic length is much smaller than the typical scale of density variation, the beam behaves like entering an infinite plasma, and the results obtained for an infinite and homogenous plasmas can be applied locally.

Having made this last assumption, we can calculate the amplification at $z = Z$ of an unstable mode excited at $z = 0$ and traveling with phase velocity $V_\phi$. If the phase velocity is not aligned with the $z$ axis, we only need to consider its $z$ component since the system is homogenous in the other directions. We therefore set the phase velocity along the $z$ axis, knowing we will only consider its $z$ component if such is not the case. A mode excited at $z = 0$ and moving at $V_\phi \neq 0$ will be amplified as it penetrates the plasma (the case $V_\phi = 0$ is treated bellow). Between $z$ and $z + dz$, the mode grows by a factor of $\exp(\delta dz/V_\phi)$, $dz/V_\phi$ being the time required to cross the slice $dz$ and $\delta$ the local growth rate. The total amplification $A(Z)$ between $z = 0$ and $Z$ will be the product of all the amplifications through the slices $dz$, that is,

$$A(Z) = \prod \exp\left( \frac{\delta dz}{V_\phi} \right),$$

where the product runs across every slice $dz$ from $z = 0$ to $Z$. If we now write the product of exponentials as the exponential of the sum and have $dz \to 0$ we find

$$A(Z) = \exp[\Delta(Z)],$$

with

$$\Delta(Z) = \int_0^Z \frac{dz}{V_\phi}.$$

This is the equation we will apply to FIS conditions, and it obviously reduces to $A(Z) = \exp(\delta Z/V_\phi)$ in a homogenous medium where neither $\delta$ nor $V_\phi$ depend on $z$.

We finally need to single out the case of an absolute unstable mode with zero phase velocity. This is important for the FIS because filamentation instability has this very property. For such nonconvective unstable modes with $V_\phi = 0$, the preceding analysis does not apply because the mode does not move through the successive plasma slices as time passes. A filamentation mode excited at $t = 0$ and $z = Z$ with an initial amplitude $A_0(k, z)$ just grows like

$$A_0(k, z) \exp(\delta t),$$

until it reaches the nonlinear stage.

### III. FAST IGNITION SCENARIO

The FIS involves the transport of a relativistic electron beam from the outer region of the pellet with $n_e \sim 10^{21} \text{ cm}^{-3}$ to the DT core at $n_e \sim 10^{25} \text{ cm}^{-3}$. If we assume an exponential plasma density profile between these points $n_e = n_0 \exp(z/\lambda)$ (Ref. 16) such as the one displayed on Fig. 1, the typical length of the density gradient is given by

$$\frac{n_e}{\partial n_e/\partial z} = \lambda.$$  

We set $n_0(0) = n_0 = 10^{21} \text{ cm}^{-3}$ and define $Z_c$ as the point where the highest density is reached, with $n_e(Z_c) = n_e = 10^{25} \text{ cm}^{-3}$. Considering $Z_c \sim 100 \mu m$, we find

$$\lambda = Z_c/\ln 10 = 10.85 \mu m.$$  

It is worth noticing here that due to the ln function, this length is not very sensitive to the density magnification between 0 and $Z_c$, a $10^5$ factor instead of $10^4$ yielding, for example, $\lambda = Z_c/\ln 10^4 = 8.68 \mu m$.

We evaluate now the behavior of the unstable modes along the density gradient. It is indeed very instructive to study the most unstable one plus two others. These three modes are the one yielding the maximum two-stream growth rate with wave vector $k_{TS} = (k_{\perp, TS}, 0)$, the mode yielding the maximum filamentation growth rate with $k_{F} = (0, k_{\parallel})$, and finally the two-stream/filamentation mode, which is the fastest growing one for a relativistic beam and has $k_{TSF} = (k_{\perp, TS}, k_{\parallel})$. Remarkably, this last wave vector is a simple combination of the others. As previously noticed, we need $L_{TS} = 1/k_{\perp, TS} \ll \lambda$ to locally apply results obtained without density gradient. This scale length is almost the skin depth of the plasma since $L_{TS} \sim V_\phi/\omega_p \sim c/\omega_p$. With an initial plasma density of $n_0 = 10^{21} \text{ cm}^{-3}$, we find
We can see here a very important result: the condition $L_{TS} \ll \lambda$ is largely satisfied at the beginning of the beam path, and will be even more valid as it penetrates into the plasma. One could argue that the medium is inhomogeneous in the direction normal to the beam since a fusion pellet is spherical. According to the size of the beam, $1/k_F^b$ should be compared to the pellet radius $R$, or the beam size. If compared with the pellet radius, the inequality $1/k_F^b \ll R$ is amply satisfied because $1/k_F^b \ll 1/L_{TS} \ll R$. If compared to the beam size $L_{b,i}$, the same inequality will hold as well since simulations show $L_{b,i} \sim 20 \mu m$. We can therefore locally apply results obtained for an infinite plasma from $z=0$ to $Z_c$.

We now write the growth rate of the three instabilities with the first-order temperature corrections for a collisionless plasma and nonrelativistic beam and plasma temperatures. As previously mentioned, there is a continuum of unstable modes so that there is not one two-stream mode, nor there is one filamentation mode. But when speaking about the “two-stream growth rate,” for example, we actually mean “the maximum growth rate found on the two-stream profile.” With this in mind, growth rates for the two-stream (TS), the filamentation, and the two-stream/filamentation (TSF) instabilities are$^4$13

$$\delta_{TS} = \omega_p \sqrt{\frac{3}{2}} \frac{\alpha^{1/3}}{\gamma_b},$$

$$\delta_{TSF} = \omega_p \sqrt{\frac{3}{2}} \left( \frac{\alpha}{\gamma_b} \right)^{1/3},$$

$$\delta_F = \omega_p \beta \sqrt{\frac{\alpha}{\gamma_b}} \left( 1 - \frac{\rho_b}{\sqrt{\alpha} \gamma_b} \right).$$

(9)

It is readily seen in the previous equations that the largest growth rate for a relativistic beam is the TSF one, in view of its $1/\gamma_b^{1/3}$ scaling. Turning to the $z$ component of the phase velocity for these three modes, we know that it is equal to the beam velocity for the TS and TSF modes,$^13$ whereas, as previously mentioned, it vanishes for the filamentation instability. We therefore use Eq. (4) for TS and TSF and Eq. (5) for filamentation to compute the gradient effect on the three modes.

We conduct a firsthand estimate of the quadratures considering a constant beam velocity during its travel. This is quite reasonable if it is supposed to reach the core with a relativistic energy. We also keep a constant beam density, which looks natural if the beam velocity remains constant, provided it gets transmitted with a constant density from $z=0$. We do not need to make assumptions about the plasma temperature since this parameter does not appear in Eqs. (9).

However, we assume that the beam temperature is constant, which means that the beam does not have time to thermalize with itself or the plasma on its way to the dense core. The growth rates are therefore varying only under the variation of plasma electronic density and Eq. (4) can be readily integrated from $z=0$ to $Z<Z_c$.

$$\Delta_{TS}(Z) = \delta_{TS}(0) \frac{6\lambda}{V_b} \left[ \exp\left( \frac{Z}{6\lambda} \right) - 1 \right].$$

$$\Delta_{TSF}(Z) = \delta_{TSF}(0) \frac{6\lambda}{V_b} \left[ \exp\left( \frac{Z}{6\lambda} \right) - 1 \right].$$

(10)

As far as filamentation instability is concerned, Eqs. (5) and (9) simply yield

$$A_F(Z) = \exp \left[ \omega_p \beta \sqrt{\frac{\alpha}{\gamma_b}} \left( 1 - \frac{\rho_b}{\sqrt{\alpha} \gamma_b} \right) t \right].$$

(11)

But which time $t$ should we consider here? When dealing with the others instabilities, we computed the amplification at $Z$ of a mode emitted from $z=0$. Calculating the amplification at $Z$ means therefore calculating it at $t \sim Z/V_b$. We could consider the same time and replace it in the equation above. However, we do not need to specify the time here because what is really relevant is whether or not modes are exponentially growing. In this regard, it turns out that the growth rate sign changes for $\rho_b=\sqrt{\alpha(z) \gamma_b}$, as can be seen on Eq. (11). This condition defines a position instead of a time. Since $\alpha(z)=n_e/n_e(z)$ is decreasing with $z$, filamentation instability is damped beyond a given $z$. The position $Z_F$ beyond which filamentation growth rate becomes negative is

$$Z_F = \lambda \ln \left( \frac{n_b \gamma_b}{n_0 \rho_b} \right).$$

(12)

Setting $\gamma_b = 5$, $n_b/n_0 = 0.1$, and $\rho_b = 0.1$ yields

$$Z_F = \lambda \ln 50 \sim 3.9 \lambda \Leftrightarrow Z_F \sim 0.42 Z_c.$$  

(13)

This shows that the filamentation instability does not develop in the core due to its damping mechanism. Thanks to the ln function, this order of magnitude is not very sensitive to the numerical values of the various parameters.

Regarding the two convective instabilities, the formulas obtained for the TS and TSF modes are very similar since the two growth rates only differ by their $\gamma_b$ scaling. Considering the numerical values previously mentioned for the plasma, namely, $n_0=10^{21} cm^{-3}$, and a 2 MeV beam energy yielding $\gamma_b=5$, we find that the prefactors $\delta(0)6\lambda/V_b$ in Eqs. (10) are 24.7 for TS and 72.2 for TSF, respectively. This means that if we define the $e$ folding length $Z_e$ such as $\Delta(Z_e)=1$, both modes have $Z_e \ll 6\lambda$. For $Z \ll 6\lambda$, we therefore expand the exponential and write

$$\Delta_{TS}(Z) \sim \delta_{TS}(0) \frac{Z}{V_b},$$

$$\Delta_{TSF}(Z) \sim \delta_{TSF}(0) \frac{Z}{V_b}.$$  

(14)

These formulas no longer mention the gradient scale $\lambda$, which means that these modes are amplified by a factor $e$ even before they “feel” the gradient. Still with $n_0=10^{21} cm^{-3}$, $n_b/n_0=0.1$, and $\gamma_b=5$, the $e$ folding distances are, respectively, 2.58 $\mu m$ for TS and 0.88 $\mu m$ for TSF so that both are smaller than the gradient scale of $\lambda = 10.85 \mu m$. 


IV. DISCUSSION AND CONCLUSION

We have shown that a WKB-like approximation is applicable in the FIS even with an exponentially growing plasma density because the plasma skin depth always remains smaller than the typical distance of the gradient. We restricted to collisionless plasmas and nonrelativistic beam and plasma temperatures. Using this approximation to investigate the amplification of the three key modes involved in the relativistic beam/plasma interaction physics, we find two opposite gradient effects. On one hand, the filamentation instability is reduced in an increasing manner as it enters the plasma because of transverse beam temperature. In typical FIS conditions, the system is no longer filamentation (F) unstable by the time it is half-way to the core. On the other hand, the TS and the TSF instabilities do not have any efficient thermal damping mechanism and both grow significantly even before they feel the density gradient. Considering the core is 100 μm distant from the starting point of the beam, TS is amplified 6 times after 2.58 μm and TSF after only 0.88 μm.

It is important to notice at this stage that, in reality, every instability is growing (or not) at the same time. Even if the fastest growing mode saturates first, it is difficult to talk about the saturation time of any given mode because it depends on its initial amplitude, which in turns depends on the noise, or the defects, which prompted the instability in the first place. But once this has happened, one can no longer consider that the slower modes keep growing in their linear regime because the “ground state” of the beam has changed. In other words, filamentation instability, for example, is the growth of a given mode perturbating an equilibrium ground state consisting in two homogenous beams going opposite directions. As previously said, it is damped beyond Zr given by Eq. (12). But when the beam arrives at Zp, this equilibrium ground state will certainly have been destroyed for long by the growth of the faster instabilities. Filamentation would keep growing the way we described here if it were the only instability of the system. And this is true for every unstable mode. It is therefore unrealistic to talk about the evolution of the linear regime of a given mode once the fastest instabilities have saturated. On the other hand, it is very important to evaluate the density gradient effects on the evolution of the system while every mode is in its linear phase in order to guess what the basis of the nonlinear regime will be.

Lastly, it would be worth studying in details the synchronism between the wave vector of a convective mode traveling into the plasma and the maximum growth rate at the position considered. To be clear on this point, we schematically picture on Fig. 2 the TS growth rate curves in terms of the parallel wave number for various plasma densities at \( z_1 < z_2 < z_3 \). The wave vector yielding the maximum growth rate is \( \sim \omega_p/\sqrt{V_b} \approx n_s(z)_{1/2} \), and the maximum growth rate itself is scaled like \( n_s(z)_{1/6} \). Concerning the much less investigated TSF modes, their maximum growth rate is also scaled like \( n_s(z)_{1/6} \) and is reached for the same parallel wave vector. Therefore, as displayed on Fig. 2, the growth rate curve shifts to the wider curves when the beam progresses through the plasma (from the \( z_1 \) to the \( z_3 \) curve). If we then consider a mode with wave vector \( k_0 \) emitted from \( z_1 \) and a growth rate which is maximum at that point, it is clear that \( k_0 \) no longer corresponds to the maximum growth rate when the wave has entered in the plasma. It is nevertheless possible to neglect this effect in first approximation for two reasons. Firstly, the wave vector is shifting to the “good side” of the growth rate curve. The growth rate variation would be very sharp if the curve was shifting in such a way that \( k_0 \) is moving towards the maximum unstable wave vector \( k_m \) \( k_m \) reads \( k_m \sim k_0(1 + \Delta k) \) with \( \Delta k = n_s^{1/6}/\gamma_b \). But the variation is much softer because \( k_0 \) is shifted towards the softer side of the curve. Secondly, the growth rate near its maximum is scaled like \( n_s(z)_{1/6} \) so that plasma density variations are soften by the 1/6 power. We can therefore consider that the growth rate corresponding to \( k_0(0) \) keeps sticking to its maximum value for \( z > 0 \), especially if we consider the beginning of the path as in Eqs. (14).

Also in connection with the synchronization between a given wave vector and the maximum growth rate, we need to stress that the preceding discussion assumes that the wave vector remains constant during the propagation. However, Breizman and Ryutov expressed the variation of the parallel wave vector along the plasma density gradient as

\[
\frac{dk_0}{dt} = -\frac{d\omega_p}{dz} = -\frac{\omega_p(0)}{2\lambda} \exp\left(\frac{z}{2\lambda}\right),
\]

so that

\[
\frac{dk_0}{k_0(0)} = -\frac{dz}{2\lambda} \exp\left(\frac{z}{2\lambda}\right) = \frac{\Delta k}{k_0(0)} = 1 - \exp\left(\frac{z}{2\lambda}\right).
\]

It is found that this effect adds up to the other one since it also contributes to a shifting of the original wave vector to the left side of the growth rate curve. We also find that the relative variation remains small as long as \( z < 2\lambda \).

With some e folding distances of \( \lambda/4.2 \) for TS and \( \lambda/12.3 \) for TSF, it is almost certain that TSF should leave the linear regime before these effects become important (TSF amplification by the time it reaches \( \lambda \) is \( e^{12.3} = 2.2 \times 10^5 \)). Even if TS may not saturate before it reaches \( \lambda \) (TS amplification in “only” \( e^{4.2} = 67 \) when it reaches \( \lambda \)), the system will have already left the linear regime through the early saturation of TSF, thus also leaving the validity domain of the present analysis. The real issue is therefore to evaluate the fastest growing mode in the gradient, as well as the way it
should saturate. We here considered a scenario where the beam density is smaller than the plasma one from the very beginning. In such a situation, growth rates are given by Eqs. (9) and TSF domination is reinforced by the gradient. Such a scenario therefore prompts the need for a nonlinear investigation of TSF evolution while doing the same for TS is less important and for F irrelevant. But other scenarios with larger beam densities should also be considered for FIS,7 and this can dramatically affect the order of precedence of growth rates at the beginning of the density gradient.19

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