Mixed two-stream filamentation modes in a collisional plasma

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I. INTRODUCTION

The interaction of a relativistic electron beam with a plasma is relevant, among others, for fast ignition scenario\textsuperscript{1} where the precompressed deuterium-tritium (DT) core of a fusion target is to be ignited by a laser-generated relativistic electron beam. The relativistic electron beam quickly prompts a return current so that one eventually has to deal with a typical two-stream configuration which is subjected to various electromagnetic instabilities. Much effort have been devoted in the last years to investigate these instabilities,\textsuperscript{2–10} whether it be the two-stream or the filamentation (also sometimes referred as Weibel\textsuperscript{2,4,11}) instabilities. These instabilities are usually analyzed by linearizing the relativistic Vlasov (or fluid) and Maxwell equations. Then, the response of the linearized equation to a perturbation \( \propto \exp(ik \cdot r - i\omega t) \) is investigated and one eventually finds some unstable self-excited modes. At this stage, the orientation of the wave vector \( k \) plays an important role. Choosing a wave vector parallel to the beam velocity \( V_b \) yields the two-stream unstable modes which happen to be purely longitudinal. On the other hand, choosing a wave vector normal to the beam yields the purely transverse filamentation unstable modes. The exploration of the much less investigated intermediate orientations has brought a very important result by that showing that the strongest instability suffered by the system is eventually to be found for an oblique wave vector.\textsuperscript{6–9,12} This most unstable mode is a mixture of the two-stream and the filamentation instabilities but is not damped as the last two can be. For example, the maximum two-stream growth rate is reduced by a factor of 1/\( \gamma_b \) in the relativistic regime, while the most unstable mode only decreases by a factor of 1/\( \gamma_b^{1/3} \) where \( \gamma_b \) is the beam relativistic factor. The filamentation growth rate varies like 1/\( \gamma_b^{1/2} \) and may be reduced, even canceled, by a transverse beam temperature\textsuperscript{7} whereas the most unstable mode is quite unsensitive to temperatures as long as they are nonrelativistic.\textsuperscript{7} It is therefore important to check whether or not collisions may damp this mode.

When studying collisions on instabilities, two approaches are usually employed. The first one consists in “looking in one direction” in the \( k \) space and investigating every unstable modes found in this direction. For example, this approach used in Ref. 10 for collision effects on filamentation instability and new unstable modes may be discovered as a result. Another approach, and this is the one we choose here, consists in sticking to a given mode, regardless of the wave-vector orientation, and studying the collision effects.\textsuperscript{13–15} The purpose of this paper is therefore to study the collision effects, neither on two-stream nor on filamentation instabilities, but on the intermediate mode yielding the most important growth rate of the two-stream/filamentation (TSF) branch. We start introducing our formalism before we analyze the collision effects and reach our conclusions.

II. FORMALISM

We consider a homogeneous, spatially infinite, and unmagnetized plasma. The ions are assumed to form a fixed neutralizing background. We perform the usual linear analysis of Maxwell and Vlasov equations, expressing every quantities \( Q \) as \( Q = Q_0 + Q_1 \) with \( |Q_1| \ll |Q_0| \), and retaining only the first-order terms in the equations. In order to account for collisions, we add a “Krook” collision term\textsuperscript{16} to the Vlasov equation,

\[
\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} + q \left( \frac{E}{c} + \frac{v \times B}{c} \right) \cdot \frac{\partial f}{\partial p} = \nu(f_0 - f),
\]

where \( \nu \) is an effective collision frequency, \( f_0 \) the equilibrium distribution, and \( q \) the electron charge. Another collision term, referred as Fokker–Planck, can also be used for analytical purposes\textsuperscript{17–20} and reads

\[
\nu \frac{\partial}{\partial \mathbf{v}} \left( v f + \tau \frac{\partial f}{\partial \mathbf{v}} \right),
\]

where \( \tau \) is the root mean square of the equilibrium distribution \( f_0 \). This later collision term, unlike the Krook one, has
many attractive features: it conserves the number of electrons and yields the Maxwell distribution for the equilibrium state. Also, with the occurrence of a first and second derivatives of the distribution function, the Fokker–Planck collision term can be more important than the Krook one for distributions displaying sharp gradients in the velocity space. Such is the case of the δ function which we shall use in the sequel. However, calculations are very involved with this operator, even in the one-dimensional (1D) electrostatic problem. Because we need to consider the whole three-dimensional (3D) electromagnetic formalism in order to reach the mixed two-stream/filamentation mode, we rather choose the Krook term for its analytical simplicity and its ability to evidence the basic physics of collisions (see Refs. 14, 21, and 22, for example).

As far as the collision frequency \( \nu \) involved in Eq. (1) is concerned, the most general approach in the small beam density regime consists in considering a plasma electron-electron collision frequency \( \nu_{ee} \) (Refs. 14, 15, and 23). It is then possible to include the smaller plasma electron-ion collisions setting \( \nu = \nu_{ee} + \nu_{ei} \), with \( \nu_{ei} \sim \nu_{ee}/2.5 \) in the fast ignition scenario (FIS) conditions.

The next step is to derive the dielectric tensor. We therefore write the Maxwell equations before we linearize them,

\[
\text{curl } E = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t},
\]

\[
\text{curl } B = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{J}.
\]

Expanding every quantities in Fourier series and eliminating the perturbed magnetic field between these two equations yield

\[
\frac{\omega^2}{c^2} \left( \mathbf{E} + \frac{4i\pi}{\omega} \mathbf{J} \right) + \mathbf{k} \times (\mathbf{k} \times \mathbf{E}) = 0.
\]

On the other hand, the current \( \mathbf{J} \) can be expressed from the perturbed distribution function \( f_1 \). The linearized Vlasov–Krook equation yields

\[
\mathbf{J} = q \int \frac{\partial f_1(\mathbf{p})}{\partial t} d^3\mathbf{p}
\]

\[
= q^2 \int \frac{[\mathbf{E} + \frac{\mathbf{v} \omega \times (c/\omega \mathbf{k} \times \mathbf{E})}{\omega + i\nu - \mathbf{k} \cdot \mathbf{v}}] \cdot \frac{\partial f_1(\mathbf{p})}{\partial \mathbf{p}}}{\omega + i\nu - \mathbf{k} \cdot \mathbf{v}} d^3\mathbf{v}.
\]

It can be noticed that Vlasov equation alone just yields an expression of \( f_1 \) where collisions shift the frequency from \( \omega \) to \( \omega + i\nu \). Maxwell equations on the other hand remain the same with collisions and do not shift the frequency. When put together, these equations give expressions where the frequency is sometimes shifted, and sometimes not. This is why in the most general case, collision effect does not simply eventually comes down to a growth rate shift, although, as we shall check, this may be true in some special cases. Eliminating the current \( \mathbf{J} \) between Eqs. (5) and (6) yields the general form from of the dispersion equation

\[
\det \left| \frac{\omega^2}{c^2} \epsilon_{\alpha\beta} + k_\alpha k_\beta - k^2 \delta_{\alpha\beta} \right| = 0,
\]

where the dielectric tensor elements are

\[
e_{\alpha\beta}(\mathbf{k}, \omega) = \delta_{\alpha\beta} + \frac{\omega_\beta^2}{n_e^\omega} \frac{2}{\omega^2} \int \frac{p_\alpha \partial f_0}{\gamma} \frac{m \gamma \omega - \mathbf{k} \cdot \mathbf{p}}{\gamma m \gamma (\omega + i\nu) - \mathbf{k} \cdot \mathbf{p}} d^3\mathbf{p}
\]

\[
+ \frac{\omega_\alpha^2}{n_e^\omega} \frac{2}{\omega^2} \int \frac{p_\beta \partial f_0}{\gamma} \frac{\mathbf{k} \cdot \partial f_0/\partial \mathbf{p}}{\gamma m \gamma (\omega + i\nu) - \mathbf{k} \cdot \mathbf{p}} d^3\mathbf{p}.
\]

In this equation, \( \omega_p \) is the plasma electron frequency, \( n_e \) the plasma electron density, and \( m \) the electron mass. Let us remind us that we use a relativistic formalism so that \( \mathbf{p} = \gamma m \mathbf{v} \) in this equation. In the nonrelativistic collisionless limit, the first quadrature reduces to \(-\delta_{\alpha\beta} \omega_p^2 / \omega^2 \), where \( \delta \) is the Kroenecker delta symbol.

Let us comment Eq. (8) and especially the way quadratures, such as

\[
\int \frac{f(x)}{x} dx,
\]

are given a meaning. Landau first pointed out\(^{25} \) the proper way to interpret this divergent quadratures in the collisionless regime (\( \nu = 0 \)) was to start assuming a nonzero collision frequency \( \nu > 0 \). The denominator which appears in quadrature (9) then switches to \( x - i\nu \) and the integration contour is the real axis. As \( \nu \to 0 \), this is equivalent to an integration along the same real axis, bypassing the pole \( x = 0 \) through an infinitesimal semicircle below the pole. The real part of the integration contour gives the Cauchy principal part (PP) of expression (9) while the half circle gives the half residue of the integrand around \( x = 0 \). The same result can also be reached by multiplying the numerator and the denominator of the collisional integrand by \( x + i\nu \). This yields

\[
\int \frac{f(x)}{x - i\nu} dx = \int \frac{x f(x)}{x^2 + \nu^2} dx + i \int \frac{\nu f(x)}{x^2 + \nu^2} dx,
\]

where quadratures are now well defined for any \( x \) real. When \( \nu \to 0 \), this expression directly gives the Plemelj formula used by Landau

\[
\lim_{\nu \to 0} \int \frac{f(x)}{x - i\nu} dx = \text{PP} \int \frac{f(x)}{x} dx + i \pi \text{f}(0).
\]

This explains why, even in the collisionless regime, the dielectric tensor has an imaginary part leading to the Landau damping, for example. Now, if we work in the collisional regime, we do not need to let \( \nu \) tend to 0 in Eqs. (8) and (10) so that a direct calculation of the dielectric tensor elements (8) shall “naturally” yield the correct imaginary part of the dielectric tensor. If the quadratures can be calculated exactly, and they can with the distribution functions we shall use, we do not have to worry about the way to implement the Plemelj formula for a triple quadrature. The exact result automatically includes what is sometimes referred as the “Landau poles.”

We therefore consider an infinite beam of velocity \( \mathbf{V}_b \) aligned with our \( z \) axis and uniform density \( n_b \) passing through a plasma with density \( n_p \). The beam induces a return
current $V_p$ in the plasma which neutralizes the beam current so that $n_b V_b + n_p V_p = 0$. The system at equilibrium is therefore charge and current neutralized. It can be proved that as long as the equilibrium function can be expressed as $\Sigma g_1^*(u)g_2^*(v)$, the electromagnetic dispersion equation for the TSF branch reads
\begin{equation}
(\eta^2 \epsilon_{xx} - k_x^2)(\eta^2 \epsilon_{zz} - k_z^2) - (\eta^2 \epsilon_{zz} + k_x k_z)^2 = 0,
\end{equation}
where $\eta = \omega/c$ and the wave vector $k$ is defined as $k = (k_x, k_z)$. Setting $k_z = 0$ in this equation, one retrieves the dispersion equation for the two-stream instability. On the other hand, filamentation instability is retrieved through $k_z = 0$. Since the former modes are purely longitudinal whereas the latter are purely transverse, one readily sees that waves along the branch switch continuously from longitudinal to transverse. This is why neither a longitudinal nor a transverse formalism can give account of the intermediate modes (although some approximations are possible, as we shall see).

In order to progress any further in the resolution of the problem, we now need to introduce the equilibrium function $f_0$. Here again, we shall use the simplest function displaying the correct maximum growth rate for an oblique wave vector. When the beam is not relativistic, the maximum growth rate all over the TSF branch is the two-stream one. But when $V_b \ll c$, the maximum leaves the $z$ axis and “enters” into the $(k_x, k_z)$ plane because electrons are heavier to move along the beam than in oblique directions. The simplest model of the system, namely, a fluid model, yields the correct value of the maximum growth rate
\begin{equation}
\delta \omega_{m}^{TSF} \sim \omega_p \left( \frac{n_b n_p}{\gamma_b} \right)^{1/3},
\end{equation}
although this maximum is not localized on one wave vector. Figure 1 shows the growth rate map obtained when the distribution function is cold for the beam and the plasma. The dimensionless growth rate $\delta \omega_p$ is plotted in terms of the dimensionless wave vector $Z = k V_p / \omega_p$. One can notice the two-stream profile for wave vectors along the beam ($V_p \parallel z$). The two-stream instability reaches its maximum for $Z \approx 1$ and vanishes soon after. Turning to wave vectors normal to the beam velocity, one finds the filamentation instability. In a fluid model, this instability just saturates at high $Z$. Temperatures can affect this picture in many ways, but the important point is that the maximum growth rate given by Eq. (13) is weakly affected by beam or plasma temperatures as long as they are kept small, which mean nonrelativistic in the present setting of a relativistic electron beam. The behavior of the TSF growth rate for higher temperatures, namely, in the kinetic regime, is to establish, although a $n_b/n_p$ scaling similar to the two-stream instability could be conjectured. This important property of the maximum growth rate allows us to investigate the collision effects on the simplest model of a cold relativistic beam interacting with a cold plasma, knowing the results obtained for the maximum growth rate can be extended to hot systems with nonrelativistic temperatures. The explicit dispersion equation thus obtained is reported in Appendix A, and shows how even the simplest model yields some involved dispersion equation as soon as electromagnetic effects all over the $k$ space are accounted for. We now present a general analysis of the collision effects before turning to their quantitative evaluation on the relativistic fluid maximum TSF growth rate.

III. MAXIMUM GROWTH RATE ANALYSIS

Having presented the basic model of the maximum TSF growth rate, we shall now evaluate collision effects on this most unstable mode. Setting $\omega' = \omega - i \nu$, the dispersion equation (12) now reads
\begin{equation}
\left[ \eta^2 \epsilon_{xx}(k, \omega') - k_x^2 \right] \left[ \eta^2 \epsilon_{zz}(k, \omega') - k_z^2 \right] - \left[ \eta^2 \epsilon_{zz}(k, \omega') + k_x k_z \right]^2 = 0,
\end{equation}
where $\eta' = \omega'/c$. Let $\omega_0(k)$ be a complex root of the collisionless dispersion equation and $\omega_z(k)$ the corresponding root of the collisional problem, so that $\lim_{\nu \rightarrow 0} \omega_\nu(k) = \omega_0(k)$. If $\nu \ll |\omega_0|$, then one can set $\omega' \approx \omega_0 + i \nu \rightarrow \omega(k)$ in Eqs. (8) and (14) and the resulting dispersion equation for $\omega(k)$ is exactly the one for $\omega_0(k)$. The roots shall therefore be the same with $\omega'_0(k) \sim \omega_0(k)$, yielding
\begin{equation}
\omega_z(k) - \omega_0(k) \rightarrow \delta \epsilon(k) \sim \delta \epsilon_0(k) - \nu.
\end{equation}
We thus find a very simple result which is valid for a wide class of distribution functions: as long as the collision frequency is small, the effect of collisions simply consists in subtracting the collision frequency from the collisionless growth rate. However, the validity condition of this result varies with $k$ since $\omega_0$ is a function of $k$. The proper frequency $\omega_0(k)$ can be expressed through its real and imaginary parts as
\begin{equation}
\omega_0(k) = \omega_r(k) + i \omega_i(k).
\end{equation}
We know from previous studies that
\begin{equation}
\omega_0(k)/\omega_p \sim Z \cos \theta_k = Z_z,
\end{equation}
where $Z = k V_p / \omega_p$ and $\theta_k = (V_p, \mathbf{k})$. As far as the imaginary part (the growth rate) is concerned, all that can be said is that it is much smaller than the plasma frequency but there is no general analytical formula for it all over the $k$ space. We can see straightforwardly that due to the expression of $\omega_0(k)$, the range of validity of Eq. (15) shall be limited to some given region of the $k$ space. For the filamentation instability with $\theta_k = \pi/2$, the condition is very demanding and the col-

FIG. 1. (Color online). Numerical evaluation of the TSF growth rate in terms of $Z = k V_p / \omega_p$. The parameters are $n_b/n_p = 0.05$ and $\gamma_b = 4$. 

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lision frequency must be directly compared to the growth rate itself since \( \omega_0(k) \) vanishes. If we restrict to the region where the maximum growth rate is located, we set \( Z_x \sim 1 \) to obtain the real part of the proper frequency and replaces the imaginary part by Eq. (13). This yields the condition under which our approximation is valid for the maximum growth rate,

\[
\frac{\nu}{\omega_p} \ll \sqrt{1 + \frac{3}{2^{2/3}} \left( \frac{\alpha}{\gamma_b} \right)^{2/3}} \sim 1 + 0.23 \left( \frac{\alpha}{\gamma_b} \right)^{2/3}.
\]  

(18)

Interestingly, this condition is very close to the weakly collisional regime hypothesis \( \nu \ll \omega_p \), assumed for using the Vlasov equation with the Krook collisional term. Figures 2(a)–2(c) display a numerical evaluation of the growth rate in the collisional regime. We plot the collisional growth rate together with the quantity (prime plots) \( \Delta = (\delta_{a0} - \delta_b) / \nu \) which is unity when Eq. (15) is fulfilled. The result is as expected; \( \Delta \) is found almost equal to unity except near the \( Z_x \) axis (filamentation). The collision frequency is still small \( (\omega_p / 100) \) on Figs. 2(a) and 2(a′) and the growth rate reduction is hardly noticeable as Fig. 2(a) is almost identical to its collisionless counterpart Fig. 1. Then, the collision frequency in Figs. 2(b) and 2(b′) is half the maximum growth rate, which is 0.16\( \omega_p \) here. The growth rate reduction is obvious and one can even notice a stability “island” for \( Z_x \sim 0 \) and \( Z_z \leq 0.8 \) as the two-stream instability has almost completely been canceled whereas the modes having \( Z_x > 0 \) are still unstable. Indeed, the two-stream instability is very low for the long wavelengths (small \( k \)) in the relativistic regime, and these are the first modes to be completely damped by collisions. Finally, collision frequency is almost equal to the maximum growth rate on Figs. 2(c) and 2(c′) with \( \nu = 0.14 \omega_p \). Both two-stream and filamentation instabilities have been completely damped and the only remaining unstable region is found near \( Z_x \sim 1 \) and \( Z_z \approx 1 \). Numerical calculations thus confirm Eq. (15) and the system is completely stabilized for \( \nu = 0.14 \omega_p \). It is worth noticing that since the approximation is valid as long as \( \nu \ll \omega_p \), Eq. (15) is eventually in very good agreement with the numerical results until the instability completely vanishes because \( \delta_m^{\text{TSF}} \ll \omega_p \).

It is possible to elaborate an analytic model of the growth rate for the fluid approximation we are studying here. Since unstable modes are quasi-longitudinal all over the \( k \) space except near the normal axis, the longitudinal approximation can be used for modes yielding the maximum TSF growth rate. The longitudinal dispersion equation for a cold relativistic beam interacting with a cold plasma including collisions with the Krook term simply reads

\[
1 - \frac{1}{(x + i \tau + aZ_x)^2} - \frac{Z_z^2 + \gamma^2 \gamma_x^2}{Z_x^2} \frac{a^3}{\gamma_b^3} = 0
\]  

(19)

with \( \alpha = n_b / n_p, \beta = k \nu_p / \omega_p, \) and \( \tau = \nu / \omega_p \). Knowing the maximum growth rate is to be found for \( Z_z \sim 1 \) (the most unstable waves shall always be the ones who, regardless of their orientation, can “fly by” an electron beam), the \( Z_x \)-dependent growth rate is calculated following the standard derivation method in the \( \alpha \ll 1 \) limit as

\[
\delta_m(Z_x) \sim \delta_m^{\text{TSF}} \left( \frac{\alpha^2 + \gamma^2 \gamma_x^2}{\gamma_b^3} \right)^{1/3} - i \nu.
\]  

(20)

This expression yields the maximum collisionless growth rate (13) when \( Z_x \approx 1 \). Also, the \( \nu \) dependence previously mentioned here is analytically retrieved for wave vectors having \( Z_x \sim 1 \), which includes wave vectors yielding the maximum growth rate.

**IV. CONCLUSION AND DISCUSSION**

We have presented an analysis of the mixed two-stream/filamentation modes in a collisional plasma. This is especially relevant because the “worst” instability suffered by the electron beam is to be found in this domain. We found that when the collision frequency \( \nu \) is much smaller than the plasma frequency, the collisional growth rate \( \delta_e \) can be obtained from the collisionless growth rate \( \delta_m \) by the simple equation \( \delta_e = \delta_m - \nu \). Since on the one hand this formula is valid for \( \nu \ll \omega_p \), and on the other hand \( \delta_m^{\text{TSF}} \ll \omega_p \), the approximation gives a very good account of numerical calculations until the instability is completely damped. Finally, the main result of this paper is that collisions succeed when non-relativistic temperatures fail, namely, in canceling the most unstable mode of the TSF branch.

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### APPENDIX: ELECTROMAGNETIC DISPERSION EQUATION

Inserting the equilibrium distribution function $f_0=f'_0+V_p^0$ into Eqs. (8) and (12) with $f'_0=n_p \delta(p_c) \delta(p_z) \delta(p_z-Pp)$ for the plasma and $f'_b=n_b \delta(p_c) \delta(p_z) \delta(p_z-Pb)$ for the beam yields the following electromagnetic dispersion equation:

$$0 = - \left[ \frac{Z_z Z_x}{\beta^2} + \frac{\alpha Z_x}{x + i \tau + \alpha Z_c} - \frac{\alpha Z_x}{\gamma_0(x + i \tau - Z_c)} \right]^2 + \left[ \frac{Z_z^2}{\beta^2} + x^2 - \frac{x + \alpha Z_c}{x + i \tau + \alpha Z_c} + \frac{\alpha (Z_x - x)}{\gamma_0(x + i \tau - Z_c)} \right] \times \left[ \frac{Z_z^2}{\beta^2} + x^2 - \frac{x^2 + \alpha Z_c^2 + i x \tau}{x + i \tau + \alpha Z_c} - \frac{\alpha x^2 + \gamma_0^2 Z_c^2 + i x \tau}{\gamma_0^2 (x - Z_c + i \tau)^2} \right]$$

(A1)

where $P_p=mV_p$ is the plasma return current impulsion and $P_b=m\gamma_0 V_b$ the beam impulsion. In deriving this equation, we use the relation $n_p V_p=n_b V_b$. The following dimensionless variables are also defined:

$$Z = \frac{k V_b}{\omega_p}, \quad \alpha = \frac{n_b}{n_p}, \quad \beta = \frac{V_b}{c}, \quad x = \frac{\omega}{\omega_p}, \quad \tau = \frac{\nu}{\omega_p}. \quad \text{(A2)}$$