Mechanics of Rigid Body 1.B

Kinetics, Dynamics

1.- Introduction

2.- Kinematics. Types of Rigid Body Motion:
   - Translation,
   - Rotation
   - General Plane Motion

   - Angular Momentum and Moment of Inertia
   - Fundamental Equations of Dynamics

Mechanics of Rigid Body 1.B

Kinetics, Dynamics


Angular Momentum and Moment of Inertia
Fundamental Equations of Dynamics

The general problem is:
To determine the motion of a rigid body under the action of several external and internal forces.

The approach to solve the problem
We will consider rigid bodies as made of large numbers of particles. Then, we will apply the Newton´s Laws to the particles. In this way we will use also the energy and momentum methods.

The scope
To determine the motion of rigid bodies in rotation about a fixed axis and in general plane motion, with symmetry respect a reference plane.
Mechanics of Rigid Body. Dynamics

MOTION OF THE MASS CENTER

The Internal and External forces acting on the particles that made the rigid body will be the cause of motion (its change).

Fundamental equations to describe the motion

Considering the system of particles and applying Newton’s Second Law, we can obtain

\[
\text{Particle } i \quad \mathbf{F}_{\text{ext},i} + \mathbf{F}_{\text{int},i} = m_i \mathbf{a}_i
\]

\[
\text{Particle } j \quad \mathbf{F}_{\text{ext},j} + \mathbf{F}_{\text{int},j} = m_j \mathbf{a}_j
\]

Let us examine the internal forces. These forces occur in pairs. According Newton’s Third Law, this forces are equal and opposite and have the same line of action. Considering the hold system and adding all the internal forces of the system, we can obtain

\[
\sum_{j=1}^{n} \mathbf{F}_{\text{ext},j} + \sum_{j=1}^{n} \mathbf{F}_{\text{int},j} = \sum_{j=1}^{n} m_j \mathbf{a}_j
\]

\[
y \sum_{j=1}^{n} \mathbf{F}_{\text{int},j} = \sum_{j=1}^{n} \sum_{i=1}^{n} \mathbf{f}_{ij} = 0
\]

Remembering the definition of the center of mass, we can conclude

\[
\sum \mathbf{F}_{\text{ext}} = m \mathbf{a}_{CM}
\]

♣ Only external forces can impart change of motion of the mass center.
♣ The rotational motion is not described by this equation. Only in the case of pure motion of translation, this equation provides the total description of body motion, because in this case all particles of the body have the same acceleration.
Newton’s Second Law for Angular Motion (Rotation)

**Rotation about a fixed axis. Angular Momentum and Moment of Inertia**

We need to introduce the **Angular Momentum** magnitude, about a point $O$. First, for one particle, $P$:

$$\overrightarrow{L}_{P/O} = \overrightarrow{r}_P \wedge m_p \overrightarrow{v}_P = \overrightarrow{r}_P \wedge m_p (\overrightarrow{\omega} \wedge \overrightarrow{r}_P)$$

$$|\overrightarrow{L}_p| = m_p \ r^2_p \ \sin(\phi);$$

$z$ component of $\overrightarrow{L}_P$:

$$\overrightarrow{L}_P \bullet \overrightarrow{k} = m_p \ r^2_p \ \sin(\phi). \sin(\phi) = m_p R^2 P \ \omega$$

And, for the hold body:

$$\overrightarrow{L} = \sum_p \overrightarrow{L}_p = \sum_p \overrightarrow{r}_p \wedge m_p (\overrightarrow{\omega} \wedge \overrightarrow{r}_p), \quad \text{or} \quad \oint d\overrightarrow{L} = \oint \overrightarrow{r} \wedge dm (\overrightarrow{\omega} \wedge \overrightarrow{r})$$

In some relevant special cases, depending on the mass spatial distribution around the rotation axis, it happens that the angular momentum about a point $O$ is parallel to the angular velocity,

$$\overrightarrow{L}_o = \oint \overrightarrow{dL} = \oint \overrightarrow{r} \wedge dm (\overrightarrow{\omega} \wedge \overrightarrow{r}) = \overrightarrow{\omega} \left( \int R^2 \ dm \right) = I \ \overrightarrow{\omega}$$

Where, the **Moment of Inertia** of the rigid body for the fixed rotation axis, is defined as,

$$I = \sum_j R^2_j \ m_j = \int R^2 \ dm$$

$R$ is the minimum distance from each elementary particle to the rotation axis.
NEWTON’S SECOND LAW FOR ANGULAR MOTION (ROTATION)

\[ \vec{L} = \sum_k \vec{r}_k \wedge \vec{p}_k \]

There is a general relationship for a system of particles between the time rate of angular momentum and the net torque of the external forces applied on the system. The internal forces do no change the angular momentum

\[ \frac{d\vec{L}}{dt} = \sum_k \frac{d(\vec{r}_k \wedge \vec{p}_k)}{dt} = \sum_k \vec{r}_k \wedge \frac{d\vec{p}_k}{dt} = \sum_k \vec{r}_k \wedge (\vec{F}_{\text{ext},k} + \sum_j \vec{f}_{jk}) = \sum_k \vec{r}_k \wedge \vec{F}_{\text{ext},k} \]

The net external torque acting on a system equals the rate of change of the angular momentum of the system. It is required that the point about which \( \vec{L} \) and \( \tau \) are computed must be a fixed point of an inertial reference system.

In some relevant special cases, depending of the mass spatial distribution around the rotation axis, we can find

\[ \vec{L} = I \vec{\omega} \]

the MOMENT of INERTIA

\[ I = \int R^2 \, dm \]

\[ \frac{d\vec{L}}{dt} = \frac{d(I \vec{\omega})}{dt} = I \frac{d\vec{\omega}}{dt} = I \vec{\alpha} = \vec{\tau} \]
NEWTON’S SECOND LAW FOR ANGULAR MOTION (ROTATION)

**ANGULAR MOMENTUM**

\[ \vec{L} = \sum r_k \wedge p_k \]

**SUMMARY**

This relationship holds for some relevant special cases, depending on the mass spatial distribution around the rotation axis.

\[ \vec{L} = I \vec{\omega} \]

\( I = \int R^2 \, dm \)

is the **MOMENT of INERTIA** of a rigid body for a fixed rotation axis, \( R \) is the minimum distance from each elementary particle to the rotation axis.

**NEWTON’S SECOND LAW FOR ANGULAR MOTION (ROTATION)**

The net external torque acting on a system equals the rate of change of the angular momentum of the system.

\[ \frac{d\vec{L}}{dt} = \tau \]

\[ \frac{dL_{CM}}{dt} = \tau_{CM} \]

Combining the Newton’s second law for rotation and the relationship between angular momentum and angular velocity.

\[ \tau_{CM} = I_{CM} \alpha \]

It is required that the point about which it is computed \( L \) and \( \tau \), it is a fixed point of an inertial reference system. In the case of mass center this relationship always holds.
Mechanics of Rigid Body. Dynamics

ABOUT THE RELATIONSHIP BETWEEN ANGULAR MOMENTUM, MOMENT OF INERTIA AND ANGULAR VELOCITY

When this relationship is held?:
- For ANY rigid body there always is a system of three mutually perpendicular axes, called principal axes of inertia. When the solid is rotating about one of these axes, this relationship is held.
- For tri-dimensional or bi-dimensional solid, when the solid exhibits symmetry around the axis of rotation.
- For plane slab, when the axis of rotation is perpendicular to the slab
- In the case of plane motion of rigid bodies which are symmetrical with respect to the reference plane.

What does happen when this relationship is held?
In the case of a motion of constant angular velocity, there is no applied external torque. No vibrations. Static and dynamics equilibrium

What does happen when this relationship does not hold?
The angular momentum rotates with the body about the axis, and then an external torque is required to be applied. Vibrations appear in the axis supports.

**ANGULAR MOMENTUM about an axis.** If \( z \) is the axis of rotation

\[
\text{component of } L_\text{p} \cdot \hat{k} = m_p \ r_p^2 \ \sin(\phi) \ \sin(\phi) = m_p R_p^2 \ \omega
\]

then

\[
L_z = \left( \int R^2 \ dm \right) \omega = I \ \omega \quad \text{always for whatever body} \]
EQUATIONS OF MOTION OF A RIGID BODY

\[ \sum \vec{F}_{\text{ext}} = m \vec{a}_{CM} \]  
Translation

\[ \tau_{CM} = I_{CM} \vec{\alpha} \]  
Rotation. **Warning!!**: This equation only holds in some relevant special cases.

The **mass** of a body is a measure of the inertial resistance of the body to change in its translational motion

When the motion is a pure rotation only is required the equation

The **moment of Inertia about an axis** is a measure of the inertial resistance of the object to change in its rotational motion about the axis

How to apply these equations in a plane motion

1. Draw the free-body diagram (of each body)
2. Apply the equation of motion:

Select one adequate reference system- Kinematics relationships- Calculating torques (or Moment of forces about a point) Computing cross-product- Calculating Moment of Inertia. The parallel-axis theorem.

\[ \tau_{CM} = \sum_i M_{CM} (F_{ext})_i \]  
Is the net torque of all external forces acting on the rigid body about the mass center

\[ \vec{\tau}_O = I_O \vec{\alpha} \]  
General plane motion = Translation + Rotation
Draw the angular momentum of the system about the point C and about the axis of rotation when the systems rotate clockwise and counterclockwise. Explain which of the systems the angular momentum is parallel to the angular velocity and calculate the moment of inertia.

Calculate the angular acceleration of the systems when is applied a couple $\mathbf{M}$ of magnitude 6 N.m, in the direction of axis of rotation. The mass of each sphere is 10 kg and the distance to axis is 0.5 m.
Gyroscope
The gyroscope is an example of motion in which the axis of rotation changes direction.

Conservation of Angular Momentum
When the net external torque acting on a system remains zero, we have

\[
\frac{d\vec{L}_{sys}}{dt} = 0
\]

or

\[
\vec{L}_{sys} = const
\]

This equation is the statement of the law of conservation of angular momentum

Stability of helicopter, spinning skater, spinning dancer, diver,…
Mechanics of Rigid Body. DYNAMICS Calculating the torque (moment of a force) and the angular momentum. The cross product

The Cross Product

Torque is expressed mathematically as the cross product (or vector product) of $\vec{r}$ and $\vec{F}$:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

The cross product of two vectors $\vec{A}$ and $\vec{B}$ is defined to be a vector $\vec{C} = \vec{A} \times \vec{B}$ whose magnitude equals the area of the parallelogram formed by the two vectors (Figure 10-5). The vector $\vec{C}$ is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$ in the direction given by the right-hand rule, that is, as the fingers curl from the direction of $\vec{A}$ toward the direction of $\vec{B}$ (Figure 10-6). If $\phi$ is the angle between the two vectors and $\hat{n}$ is a unit vector that is perpendicular to each vector in the direction of $\vec{C}$, the cross product of $\vec{A}$ and $\vec{B}$ is

$$\vec{A} \times \vec{B} = AB \sin \phi \hat{n}$$

**Definition—Cross Product**

If $\vec{A}$ and $\vec{B}$ are parallel, $\vec{A} \times \vec{B}$ is zero. It follows from the definition of the cross product that

$$\vec{A} \times \vec{A} = 0$$

and

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$
**Calculating the torque/angular momentum**

The cross product

![The cross product](image)

\[ \vec{C} = \vec{A} \times \vec{B} \]

**Figure 10-4** The cross product \( \vec{A} \times \vec{B} \) is a vector \( \vec{C} \) that is perpendicular to both \( \vec{A} \) and \( \vec{B} \) and has a magnitude \( AB \sin \phi \), which equals the area of the parallelogram shown.

**Figure 10-5**

**Figure 10-6** The direction of \( \vec{A} \times \vec{B} \) is given by the right-hand rule when the fingers are rotated from the direction of \( \vec{A} \) toward \( \vec{B} \) through the angle \( \phi \).

**Torque of a force**

[SI Units] = N.m

\[ \vec{\tau}_O = \overrightarrow{MO(F)} = r \wedge \vec{F} \]

\[ \tau_O = \left| \overrightarrow{MO(F)} \right| = r F \sin(\phi) = d F \]

\[ \tau_O = \left| \overrightarrow{MO(F)} \right| = d F \]
Moment of Inertia

\[ I = \sum_i R_i^2 m_i = \int R^2 \, dm \]

[SI Units] = kg \cdot m^2

Radius of gyration, \( k \)

\[ I = \sum_i R_i^2 m_i = \int R^2 \, dm = m k^2 \]

\[ k = \sqrt{\frac{I}{m}} \]
Calculating Moment of Inertia

Four equal spheres of mass \( m \) are linked by rigid mass-less rods of length \( d \). (a) Calculate the Moment of Inertia of the system about the rotation axis (in black) in the figures 1, 2 and 3; (b) Indicate in which of the cases the angular momentum will be parallel to the rotation axis.

\[
I = \int R^2 \, dm
\]

**Uniform Disk About a Perpendicular Axis Through Its Center**  
For the case of a uniform disk, we expect that \( I \) will be smaller than \( MR^2 \) since the mass is uniformly distributed from \( r = 0 \) to \( r = R \) rather than being concentrated at \( r = R \) as it is in a hoop. In Figure 9-7, each mass element is a hoop of radius \( r \) and thickness \( dr \). The moment of inertia of any given mass element is \( r^2 \, dm \). Since the disk is uniform, mass per unit area \( \sigma \) is constant. \( \sigma = M/A \), where \( A = \pi R^2 \) is the area of the disk. Since the area of each mass element is \( dA = 2\pi r \, dr \), the mass of each element is

\[
dm = \sigma \, dA = \frac{M}{A} \, 2\pi r \, dr
\]

We thus have

\[
I = \int r^2 \, dm = \int_0^R r^2 \sigma \, 2\pi r \, dr = 2\pi \sigma \int_0^R r^3 \, dr
\]

\[
= \frac{2\pi M \, R^4}{A} = \frac{\pi M}{2\pi R^2} \, R^4 = \frac{1}{2} MR^2
\]
### Moments of Inertia of Uniform Bodies of Various Shapes

<table>
<thead>
<tr>
<th>Shape</th>
<th>Axis</th>
<th>Moment of Inertia</th>
<th>Shape</th>
<th>Axis</th>
<th>Moment of Inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thin cylindrical shell</td>
<td>about axis</td>
<td>$I = MR^2$</td>
<td>Thin rod</td>
<td>about perpendicular line through one end</td>
<td>$I = \frac{1}{3}ML^2$</td>
</tr>
<tr>
<td></td>
<td>about diameter through center</td>
<td>$I = \frac{1}{2}MR^2 + \frac{1}{12}ML^2$</td>
<td>Solid cylinder</td>
<td>about axis</td>
<td>$I = \frac{1}{3}MR^2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>about diameter through center</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Solid sphere</td>
<td>about diameter</td>
<td>$I = \frac{2}{5}MR^2$</td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Hollow cylindrical shell</td>
<td>about axis</td>
<td>$I = \frac{1}{2}MR^2$</td>
<td>Hollow cylindrical shell</td>
<td>about diameter through center</td>
<td>$I = \frac{1}{3}ML^2$</td>
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</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Solid rectangular parallelepiped</td>
<td>about axis through center perpendicular to face</td>
<td>$I = \frac{1}{12}M(a^2 + b^2)$</td>
</tr>
</tbody>
</table>

A disk is a cylinder whose length $L$ is negligible. By setting $L = 0$, the above formulas for cylinders hold for disks.
The Parallel-Axis Theorem

We can often simplify the calculation of moments of inertia for various bodies by using the parallel-axis theorem, which relates the moment of inertia about an axis through the center of mass of an object to the moment of inertia about a second, parallel axis (Figure 9-9). Let \( I_{cm} \) be the moment of inertia about an axis through the center of mass of an object of total mass \( M \), and let \( I \) be that about a parallel axis a distance \( h \) away. The parallel-axis theorem states that

\[
I = I_{cm} + Mh^2
\]

Exercise. Calculate the moment of Inertia of a sphere about an axis tangent as is shown in the figure

\[
I = I_{cm} + mh^2
\]

\[
I = (2/5) \ mR^2 + m \ R^2
\]

\[
I = (7/5) \ mR^2
\]
**KINETIC ENERGY OF A RIGID BODY IN PLANE MOTION**

General plane motion = Translation + Rotation

**Rotational Kinetic Energy**

The kinetic energy of a system of particles rotating about an axis will be

\[ K = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} (m_i R_i^2) \omega^2 = \frac{1}{2} I \omega^2 \]

\[ v_i = \omega R_i \]

**Translational Kinetic Energy**

The kinetic energy of a system of particles in a translational motion will be

\[ K = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} m v_{CM}^2 \]

**Kinetic Energy of a rigid body in plane motion**

\[ K = \frac{1}{2} m v_{CM}^2 + \frac{1}{2} I_{CM} \omega^2 \]

General plane motion = Translation + Rotation about an axis through the center of mass
The principle of work and energy for a system of particles

\[ W_{total} = \sum_i \Delta K_i = \Delta K_{sys}; \quad W_{total} = W_{ext.\ forces} + W_{int.\ forces}; \quad W_{int.\ forces}(in\ a\ rigid\ body) = 0 \]

**Work of forces acting on a rigid body**

The work of a force during a displacement of its point of application from the position 1 to position 2

\[ W_{1\rightarrow 2} = \int_1^2 F \, d\vec{r} = \int_1^2 F \cos(\alpha) ds = \int_1^2 F_T \, ds \]

The work of a couple acting on a rigid body

\[ W_{1\rightarrow 2} = \int_{\theta_1}^{\theta_2} \overrightarrow{M} \, d\theta = \int_{\theta_1}^{\theta_2} M_z \, d\theta \]

Where \( \overrightarrow{M} \) is the moment of the couple and \( M_z \) is the component of \( \overrightarrow{M} \) in the direction of the rotation axis.

If there is no rotation, the couple does not work.

**When a rigid body rolls without sliding on a fixed surface, the friction force at the point of contact does not work.**
Mechanics of Rigid Body. DYNAMICS. Energy and Work

PRINCIPLE OF WORK AND ENERGY FOR A RIGID BODY

**Power** (the time rate at which work is done)

In the case of a rigid body acted upon by an external force $\mathbf{F}$, and moving with a velocity $\mathbf{v}$, the power was expressed as

$$\text{Power} = \frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v}$$

In the case of a rigid body rotating with an angular velocity $\omega$ and acted by a couple of moment $\mathbf{M}$

$$\text{Power} = \frac{dW}{dt} = \mathbf{M} \frac{d\theta}{dt} = \mathbf{M} \cdot \omega = M_z \omega$$

Problem. Gear A has a mass of 10 kg and a radius of gyration of 200 mm; gear B has a mass of 3 kg and a radius of gyration of 80 mm. The system is a rest when a couple of magnitude 6 N.m is applied to gear B. Neglecting friction, determine (a) the number of revolutions executed by gear B before its angular velocity reaches 600 rpm [4.35 rev]. (b) the tangential force which gear A exerts on gear B [46.2 N]. (c) the power supplied to gear B [432.72 W]. (d) the work done by the couple till the gear B reaches 600 rpm [163.9 J].
1.- A bowling ball of radius R=11 cm and mass M=7.2 kg is rolling without slipping down a plane inclined at an angle 30º above the horizontal. (a) Determine the acceleration of the center of mass and the frictional force between the ball and the incline (b) If the bowling ball was released from the rest at a height of 3 m above of horizontal plane, determine its angular velocity and the speed of the mass center when the ball reaches the horizontal plan. Repeat for a disk and for a solid cylinder with the same radius and mass and compare the results. Which of these objects will take greater speed once they reach the horizontal plane?.

2.- A uniform thin rod of length L = 1 m and mass M = 10 kg, pivoted at one end, as shown in the figure, is held horizontal and then released from rest. (a) the reactions on the pivot and the angular acceleration just at the moment in which the rod is released (b) the angular velocity of the rod and the reactions on the pivot when it reaches its vertical position.

3.- An Atwood machine has two blocks with masses m₁ and m₂, connected by a string on negligible mass that passes over a pulley with frictionless bearings. The pulley is a uniform disk of mass M and radius R. Find the angular acceleration of the pulley and the linear acceleration of each block m₁ = 10 kg; m₂= 12 kg; M = 3 kg; R= 0.20 m
4.- When the forward speed of the truck shown was 15 m/s, the brakes were suddenly applied, causing all four wheels to stop rotating. It was observed that the truck skidded to rest in 20 m. Determine the magnitude of the normal reaction and of the friction force at each wheel as the truck skidded to rest. Weight of the truck: 15.000 kg

5.-A rope is wrapped around a cylinder of length \( L = 50 \text{ cm} \), radius \( r = 20 \text{ cm} \) and mass \( m = 20 \text{ kg} \). Knowing that the cylinder is released from rest, determine (a) the tension in the rope (b) the velocity of the center of the cylinder after it has moved downward a distance \( s = 2 \text{ m} \).

6.-A cycle rider maintains an upward constant speed of 5 m/s in a incline 10°. (a) Assuming no wind friction, calculate the supplied power by the cycle rider. (b) Draw the free-body diagram (b1) for the complete system, (b2) for the front wheel (b3) for the rear wheel. © Which is the angular velocity of the rear wheel if its radius is 0.35 m. Mass 70 kg

7.- The bowling ball of radius \( R = 11 \text{ cm} \) and mass \( M = 7.2 \text{ kg} \), is thrown so that the instant it touches the floor it is moving horizontally with speed 5 m/s ad is no rotating. The coefficient of kinetic friction between the ball and the floor is \( \mu = 0.08 \). Find (a) the time the ball slide and (b) the distance the ball slides before it rolls without slipping.