MECHANICS

Kinematics of Particles:

Kinematics is the study of the geometry of motion; is used to relate displacement, velocity, acceleration and time, without reference to the cause of motion.

Motion in One Dimension
Motion in Two and Three Dimensions

Basic References:
MECHANICS
Kinematics of Particles
Motion in One Dimension

Displacement, Velocity, and Speed
Position $x_i$ is defined by a frame of reference.

Displacement: The change in the position of particle, m

$$\Delta x = x_f - x_i$$

Velocity: The rate at which the position change, m/s

Average speed: The ratio of the total distance traveled to the total time from start to finish.

Instantaneous speed: The limit of the ratio $\Delta x / \Delta t$ as $\Delta t$ approaches zero.

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{ds}{dt}$$

= slope of the line tangent to the $x$-versus-$t$ curve.

Traveled distance: $s$

Average speed: $\frac{s}{t}$

Instantaneous speed: $\frac{ds}{dt}$
Average and Instantaneous velocity.  

The average velocity is the slope of the straight line connecting the points \((t_1, x_1)\) and \((t_2, x_2)\).

Geometric Interpretation of Average Velocity

The instantaneous velocity is the limit of the ratio \(\Delta x/\Delta t\) as \(\Delta t\) approaches zero.

\[
v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{ds}{dt}
\]

= slope of the line tangent to the \(x\)-versus-\(t\) curve

Definition—Instantaneous Velocity

The magnitude of the instantaneous velocity is the instantaneous speed.
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Motion in One Dimension

Acceleration

“It goes from zero to 60 in about 3 seconds.”
© Sydney Harris
**Acceleration** is the rate of the change of instantaneous velocity

\[
a_{av} = \frac{\Delta v}{\Delta t}
\]

Average acceleration, m/s\(^2\)  \hspace{1cm} 2-8

\[
a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d(dx/dt)}{dt} = \frac{d^2x}{dt^2}
\]

Instantaneous acceleration, m/s\(^2\)

\[
= \text{slope of the line tangent to the } v\text{-versus-}t \text{ curve} \hspace{1cm} 2-9
\]

Motion under constant acceleration, \(a = a_{av} = \text{constant; uniformly accelerated rectilinear motion}\)

\[
v = v_o + at
\]

\[
\Delta x = x - x_o = vt + \frac{1}{2}at^2
\]  \hspace{1cm} \text{Eliminating } t \hspace{1cm} v^2 = v_o^2 + 2a\Delta x

Practical Case: Objects in Free-fall  “falling freely under the influence of gravity only”

“Near the earth’s surface all unsupported objects fall vertically with constant acceleration (provided air resistance is negligible)  Gravity acceleration \(\sim 9.81 \text{ m/s}^2\).
How is the motion that is showed in the figures? Write the equation that describes the position and acceleration in each case.

Describe the motion as is represented in the figure. Estimate the acceleration at $t=50$ s; $t=120$ s; What is the traveled distance since the origin of time to $t = 60$ s; $t= 180$ s

A car accelerates from rest as showed in the figure. Draw a graph displaying its velocity and position versus time.
Relative Motion of two particles

A frame of reference is an extended object whose parts are at rest relative to each other.

\[ x_{B/A} = x_B - x_A \]

Relative position coordinate of B with respect to A

Relative velocity of B with respect to A

\[ v_{B/A} = v_B - v_A \]

Relative acceleration of B with respect to A

\[ a_{B/A} = a_B - a_A \]

Dependent motions
The motions of particles are linked, they are not independent

Examples: Pulleys and objects linked by inextensible strings

\[
\begin{align*}
2x_B + x_A &= \text{Constant} \\
2v_B + v_A &= 0 \\
2a_B + a_A &= 0
\end{align*}
\]
Problems

A student throws her cap straight upward with an initial speed of 14.7 m/s. (a) How long does it take it highest point; (b) What is the distance to the highest point? (c) Assuming the cup is caught at the same height from which it was released, what is the total time the cap is in flight?
A ball is thrown vertically upward from the 12-m level in an elevator shaft with an initial velocity of 18 m/s. At the same instant an open-platform elevator passes the 5-m level, moving upward with a constant velocity of 2 m/s. Determine (a) when and where the ball will hit the elevator, (b) the relative velocity of the ball with respect to the elevator when the ball

**Motion of Ball.** Since the ball has a constant acceleration, its motion is uniformly accelerated. Placing the origin $O$ of the $y$ axis at ground level and choosing its positive direction upward, we find that the initial position is $y_0 = +12 \text{ m}$, the initial velocity is $v_0 = +18 \text{ m/s}$, and the acceleration is $-9.81 \text{ m/s}^2$. Substituting these values in the equations for uniformly accelerated motion, we write

$$v_B = v_0 + at$$
$$y_B = y_0 + v_0 t + \frac{1}{2}at^2$$

Substituting $v_B = 18 - 9.81t$ and $y_B = 12 + 18t - 4.905t^2$.

**Motion of Elevator.** Since the elevator has a constant velocity, its motion is uniform. Again placing the origin $O$ at the ground level and choosing the positive direction upward, we note that $y_0 = +5 \text{ m}$ and write

$$v_E = +2 \text{ m/s}$$
$$y_E = y_0 + v_E t$$

Substituting $y_E = y_B$.

**Ball Hits Elevator.** We first note that the same time $t$ and the same origin $O$ were used in writing the equations of motion of both the ball and the elevator. We see from the figure that when the ball hits the elevator,

$$y_E = y_B$$

Substituting for $y_E$ and $y_B$ from (2) and (4) into (5), we have

$$5 + 2t = 12 + 18t - 4.905t^2$$

Solving for $t$, we get

$$t = 0.39 \text{ s}, \quad t = 3.65 \text{ s}$$

Only the root $t = 3.65 \text{ s}$ corresponds to a time after the motion has begun. Substituting this value into (4), we have

$$y_E = 5 + 2(3.65) = 12.30 \text{ m}$$

The relative velocity of the ball with respect to the elevator is

$$v_{B/E} = v_B - v_E = (18 - 9.81t) - 2 = 16 - 9.81t$$

When the ball hits the elevator at time $t = 3.65 \text{ s}$, we have

$$v_{B/E} = 16 - 9.81(3.65) = -19.81 \text{ m/s}$$

The negative sign means that the ball is observed from the elevator to be moving in the negative sense (downward).
MECHANICS
Kinematics of Particles
Motion in Two and Three Dimensions
Vectors

Addition and subtraction

Parallelogram method

Vectors are quantities with magnitude and direction that add like displacements.

Definition—Vectors
Vector rectangular components

**For vector** $\vec{A}$:

- $A_y = A \sin \theta$
- $A_x = A \cos \theta$

**For vector** $\vec{B}$:

- $B_S = B \cos \theta_2 = -B \cos \theta$

**Diagram**

(a) $A_S = A \cos \theta_1$

(b) $|B_S| = B \cos \theta_2$
Addition of vectors can be done by the head-to-tail, the parallelogram method, or analytically, using vector components.
**Summary of properties of vectors**

<table>
<thead>
<tr>
<th>Property</th>
<th>Explanation</th>
<th>Figure</th>
<th>Component representation</th>
</tr>
</thead>
</table>
| Equality          | $\vec{A} = \vec{B}$ if $|\vec{A}| = |\vec{B}|$ and their directions are the same | $\vec{A}$, $\vec{B}$ | $A_x = B_x$  
                     |                                                                              |                | $A_y = B_y$  
                     |                                                                              |                | $A_z = B_z$  |
| Addition          | $\vec{C} = \vec{A} + \vec{B}$                                             | $\vec{A}$, $\vec{B}$ | $C_x = A_x + B_x$  
                     |                                                                              |                | $C_y = A_y + B_y$  
                     |                                                                              |                | $C_z = A_z + B_z$  |
| Negative of a vector | $\vec{A} = -\vec{B}$ if $|\vec{B}| = |\vec{A}|$ and their directions are opposite | $\vec{A}$, $\vec{B}$ | $A_x = -B_x$  
                     |                                                                              |                | $A_y = -B_y$  
                     |                                                                              |                | $A_z = -B_z$  |
| Subtraction       | $\vec{C} = \vec{A} - \vec{B}$                                             | $\vec{A}$, $\vec{B}$ | $C_x = A_x - B_x$  
                     |                                                                              |                | $C_y = A_y - B_y$  
                     |                                                                              |                | $C_z = A_z - B_z$  |
| Multiplication by a scalar | $\vec{B} = s\vec{A}$ has magnitude $|\vec{B}| = |s||\vec{A}|$ and has the same direction as $\vec{A}$ if $s$ is positive or $-\vec{A}$ if $s$ is negative | $\vec{B}$, $\vec{A}$ | $B_x = sA_x$  
                     |                                                                              |                | $B_y = sA_y$  
                     |                                                                              |                | $B_z = sA_z$  |
3-3
Position, Velocity, and Acceleration
\[ \vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \frac{ds}{dt} \vec{T} = v \vec{T} \]

\[ \vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d\left(v \vec{T}\right)}{dt} \]
Rectangular components of position vector, velocity and acceleration

Curvilinear Motion

\[ \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \]

\[ \vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \]

\[ \vec{a} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k} \]
Exercises

Curvilinear Motion

A student throws her cap into the air with an initial velocity of 24.5 m/s at 36.9° above of horizontal. The path described by the cap is showed in the figure. Draw the position vector, velocity and acceleration at time $t = 0$, $t = 1$, $t = 2$ and $t = 3$ (in seconds).

Consider the motion of a pendulum bob shown in figure. Draw the position vector, the velocity and the acceleration at different times, $t_0$, $t_2$, $t_4$, $t_6$ and $t_8$.

A particle is moving in a circle with constant speed. Draw a diagram motion about it displaying the position vector, velocity and acceleration in different positions of the path.
The rate of change of a vector as observed from a moving frame of reference is, in general, different from its rate as observed from a fixed frame of reference.

**However, if the moving frame is in translation, the rate of change of a vector is the same with respect to a fixed frame and with respect to a frame in translation.**

\[
\vec{r}_B = \vec{r}_A + \vec{r}_{B/A} \\
\vec{v}_B = \vec{v}_A + \vec{v}_{B/A} \\
\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}
\]

Position of B relative to the moving frame Ax’y’z’, or the position of B relative to A

Velocity of B relative to the moving frame Ax’y’z’, or the velocity of B relative to A. This is the derivative of relative position

Acceleration of B relative to the moving frame Ax’y’z’ or the acceleration of B relative to A. This the derivative of relative velocity

The motion of B with respect to the fixed frame Oxxyz is referred to as **absolute motion**.
Motion relative to a frame in translation

\[ \vec{r}_B = \vec{r}_A + \vec{r}_{B/A} \]

\[ \vec{v}_B = \vec{v}_A + \vec{v}_{B/A} \]

\[ \vec{a}_B = \vec{a}_A + \vec{a}_{B/A} \]
SAMPLE PROBLEM 11.9

Automobile A is traveling east at the constant speed of 36 km/h. As automobile A crosses the intersection shown, automobile B starts from rest north of the intersection and moves south with a constant acceleration 1.2 m/s². Determine the position, velocity, and acceleration of B relative to A 5 s after A crosses the intersection.
**SOLUTION**

We choose $x$ and $y$ axes with origin at the intersection of the two streets with positive senses directed respectively east and north.

**Motion of Automobile $A$.** First, the speed is expressed in m/s:

$$v_A = \left(36 \frac{\text{km}}{\text{h}} \right) \left(\frac{1000 \text{ m}}{1 \text{ km}} \right) \left(\frac{1 \text{ h}}{3600 \text{ s}} \right) = 10 \text{ m/s}$$

Noting that the motion of $A$ is uniform, we write, for any time $t$,

$$a_A = 0$$
$$v_A = +10 \text{ m/s}$$
$$x_A = (x_A)_0 + v_A t = 0 + 10t$$

For $t = 5 \text{ s}$, we have

$$a_A = 0$$
$$v_A = 10 \text{ m/s}$$
$$x_A = 50 \text{ m}$$

**Motion of Automobile $B$.** We note that the motion of $B$ is uniform accelerated and write

$$a_B = -1.2 \text{ m/s}^2$$
$$v_B = (v_B)_0 + at = 0 - 1.2 t$$
$$y_B = (y_B)_0 + (v_B)_0 t + \frac{1}{2}a_B t^2 = 35 + 0 - \frac{1}{2}(1.2)t^2$$

For $t = 5 \text{ s}$, we have

$$a_B = -1.2 \text{ m/s}^2$$
$$v_B = -(1.2 \text{ m/s}^2)(5 \text{ s}) = -6 \text{ m/s}$$
$$y_B = 35 - \frac{1}{2}(1.2 \text{ m/s}^2)(5 \text{ s})^2 = 20 \text{ m}$$

**Motion of $B$ Relative to $A$.** We draw the triangle corresponding to the vector equation $\mathbf{r}_{B/A} = \mathbf{r}_A + \mathbf{r}_{B/A}$ and obtain the magnitude and direction of the position vector of $B$ relative to $A$.

$$\mathbf{r}_{B/A} = 53.9 \text{ m} \quad \alpha = 21.8^\circ$$

Proceeding in a similar fashion, we find the velocity and acceleration of $B$ relative to $A$.

$$\mathbf{v}_{B/A} = 11.66 \text{ m/s} \quad \beta = 31.0^\circ$$

$$\mathbf{a}_{B/A} = 1.2 \text{ m/s}^2$$
3-4
Special Case 1: Projectile Motion
Special Case 2: Circular Motion
Tangential and Normal Components of acceleration
\[ a = a_c = \frac{v^2}{r} \]  

CENTRIPETAL ACCELERATION

\[ a_t = \frac{dv}{dt} \]  

TANGENTIAL ACCELERATION
SAMPLE PROBLEM 11.10

A motorist is traveling on a curved section of highway of radius 2500 ft at the speed of 60 mi/h. The motorist suddenly applies the brakes, causing the automobile to slow down at a constant rate. Knowing that after 8 s the speed has been reduced to 45 mi/h, determine the acceleration of the automobile immediately after the brakes have been applied.

SOLUTION

**Tangential Component of Acceleration.** First the speeds are expressed in ft/s.

\[
60 \text{ mi/h} = \left(60 \frac{\text{mi}}{\text{h}} \right) \left(3600 \frac{\text{ft}}{\text{mi}} \right) = 88 \text{ ft/s}
\]

\[
45 \text{ mi/h} = 66 \text{ ft/s}
\]

Since the automobile slows down at a constant rate, we have

\[
a_t = \text{average } a_t = \frac{\Delta v}{\Delta t} = \frac{66 \text{ ft/s} - 88 \text{ ft/s}}{8 \text{ s}} = -2.75 \text{ ft/s}^2
\]

**Normal Component of Acceleration.** Immediately after the brakes have been applied, the speed is still 88 ft/s, and we have

\[
a_n = \frac{v^2}{\rho} = \frac{(88 \text{ ft/s})^2}{2500 \text{ ft}} = 3.10 \text{ ft/s}^2
\]

**Magnitude and Direction of Acceleration.** The magnitude and direction of the resultant \( \mathbf{a} \) of the components \( a_n \) and \( a_t \) are

\[
\tan \alpha = \frac{a_n}{a_t} = \frac{3.10 \text{ ft/s}^2}{2.75 \text{ ft/s}^2} \quad \alpha = 48.4^\circ
\]

\[
a = \frac{a_n}{\sin \alpha} = \frac{3.10 \text{ ft/s}^2}{\sin 48.4^\circ} \quad a = 4.14 \text{ ft/s}^2
\]