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BOUNDARY ELEMENT FORMULATION FOR FLEXURAL ANALYSIS OF PLATES WITH VARIABLE THICKNESS

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Abstract. *The boundary element method is already a well established numerical technique and appear to be very efficient and accurate to deal with moderate thick plates in bending in the context of several practical engineering problems. In this work the BEM formulation, based on the Kirchhoff's hypothesis, is extended to incorporate the cases of varying thickness plates. For this particular case, an appropriate form of the Betti's reciprocal work theorem has been taken to derive integral representations of displacements and bending and twisting moments, starting by assuming the structural element with varying thickness. The domain integrals, remaining in all integral equations to account for the stiffness variation, are properly handled to lead to small numbers of internal unknowns and the corresponding integral representations. As usual for this problem, only deflection integral representations related to boundary collocations are taken to write the necessary algebraic relations, written for singular and non singular points.*

1 INTRODUCTION

The boundary element method (BEM) is already a well established numerical technique to deal with an enormous number of complex engineering problems. Analysis of plate bending problems using this technique has attracted the attention of many researchers during the last years. Since 1978, when a first general direct formulation based on the Kirchhoff's hypothesis has appeared, the technique has experimented a rather large growth, being nowadays applied to several practical engineering problems. The first works discussing the use of boundary element method direct formulation, in conjunction with the Kirchhoff's theory, are due to Bezine[1] and Stern[2]. Those works, as well as several other more recent publications, have pointed out the capability of the method, taking into account its accuracy and confidence, to model plates in bending. For instance, Gou-Shu[3] and Hartmann & Zotemantel[4] have presented interesting approaches where displacement restrictions at internal points and the use of hermitian interpolations were discussed in details. Venturini & Paiva[5] have shown several alternatives to define the algebraic system of equations by placing the collocations either on the boundary or at particular outside points. Besides those works reporting the linear BEM formulation for plate bending, several other works can also be pointed out to show the use of the technique to model more complex behaviours, e.g. Chueri & Venturini[6] and Moshaiov & Vorus[7].

Although the field of applications of boundary methods is nowadays rather large, formulations to treat problem characterised by plates with varying thickness plate rarely appear. The importance of this kind of formulation is not restricted to plates that exhibit varying thickness, but can also be adopted in other situations where the plate stiffness varies over the domain, including here the orthotropic cases. The first works reporting formulations to deal with this problems are due to: Sapountzakis & Katsikadelis[8]. Besides those works, it is convenient to mention another one proposed by the where a particular BEM formulation was derived to consider zoned varying thickness plates[9].

In this work the BEM formulation based on the Kirchhoff's hypothesis, derived to deal with the varying thickness plates, has been generalised. The formulation is prepared to consider any case of plate with varying thickness, including zoned problems, where thickness or the equivalent stiffness is constant over particular zones, and the orthotropic case. An appropriate form of the Betti's reciprocal work theorem has been taken to derive the required integral representations of displacements and bending and twisting moments, starting by assuming that the plate domain is characterised by exhibiting varying thickness or stiffness. According to the way chosen to derived the integral equations, displacements, rotations or moments at internal and boundary points can be left as variables inside domain integrals. Those domain integrals represent the necessary terms to account for the stiffness variation. They can be properly handled to lead to small numbers of internal unknowns and consequently requiring less corresponding integral representations.

Regarding to the algebraic system definition, as usual for the plate bending problem, only deflection integral representations are taken to write the necessary algebraic relations. Thus, as two representations are required for each node resulting from the discretization, one must write as well complementary integral equations for outside collocations. In addition, new integral representation have also to be added to the system due to the extra values left inside the domain integrals. If only displacements are left in the domain integral term, only one extra equation is necessary to be written for each internal node. Other appropriate relations must be written for internal nodes if rotations or bending and twisting moments are values remained in the domain integral terms. For any case, these domain integrals are evaluated by approaching only displacements and plate stiffness over internal cells. Classical numerical examples are used to verify the accuracy of the proposed formulation.

2 BASIC EQUATIONS

A flat plate of thickness *h,* referred to a Cartesian system of co-ordinates with axes x_1 and x_2 lying on its middle surface and axis x_3 perpendicular to that plane, is considered. It is assumed that the plate supports a distributed load g acting in the x_3 direction on the plate middle plane, in absence of any distributed external moments. For this plate, the equilibrium equations are given by the following expressions:

$$m_{ij},_j - q_i = 0 \quad \text{and} \quad q_i,_i + g = 0 \quad (1a, b)$$

where m_{ij} are bending and twisting moments, while q_i represents shear forces, with subscripts taken in the range {1, 2}.

Equations (1) can be joined together to give the well known plate bending differential equation:

$$m_{ij,ij} + g = 0 \quad (2)$$

The plate domain is denoted by Ω, while its boundary is represented by Γ. Over Γ, the following boundary conditions may be assumed: $u_i = \overline{u}_i$ on Γ_1 (deflections and rotations) and $p_i = \overline{p}_i$ on Γ_2 (normal bending moment and effective boundary shear forces), where $\Gamma_1 \cup \Gamma_2 = \Gamma$.

As usual, the generalised stress × displacement relations can be expressed in the following form:

$$m_{ij} = -D\left(\nu\delta_{ij} w,_{kkj} + (1-\nu) w,_{ijj}\right) \quad q_i = -Dw,_{jji} \quad (3a, b)$$

where $D = Eh^3 / (1-\nu^2)$ is the flexural rigidity.

Remembering that due to the Kirchhoff hypothesis, it is necessary to define the effective shear force given by,

$$V_n = q_n \partial m_{ns} / \partial x_s \quad (4)$$

where (n, s) are the local co-ordinate system, with n and s are referred to the normal and tangential directions respectively.

By replacing equation (3a) into the differential equation and assuming constant stiffness D, ones has,

$$w,_{iijj} = \frac{g}{D} \qquad (i, j = 1,2) \qquad (5)$$

where $w,_{iijj} = \nabla^2 w$, the Laplacian operator applied on the deflections.

For the case to be seen in this work the stiffness is assumed varying over the plate domain, therefore equation (5) is modified due to the presence of derivatives of D(s). As shown in Timoshenko & Krieger[10], for this case, the differential equation written in its explicit form is,

$$D\nabla^2\nabla^2 w + 2\frac{\partial D}{\partial x_1}\frac{\partial}{\partial x_1}\nabla^2 w + 2\frac{\partial D}{\partial x_2}\frac{\partial}{\partial x_2}\nabla^2 w + \nabla^2 D\nabla^2 w$$

$$-(1-\nu)\left(\frac{\partial^2 D}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2} - 2\frac{\partial^2 D}{\partial x_1 \partial x_2}\frac{\partial^2 w}{\partial x_1 \partial x_2} + \frac{\partial^2 D}{\partial x_2^2}\frac{\partial^2 w}{\partial x_1^2}\right) = g \quad (6)$$

3 INTEGRAL REPRESENTATIONS

As it is well known the direct formulation of the boundary element method for Kirchhoff plates can be derived from a reciprocity relation written in terms of bending moments and curvatures of two independent mechanical problems. For a domain Ω inserted into an infinite space, one can define curvatures $w,_{ij}(q)$ and moments $m_{ij}(q)$ at a field point q. Assuming that the corresponding well known fundamental values, $w^*,_{ij}(s,q)$ and $m^*_{ij}(s,q)$, computed for a source point s defined in the infinite space, Ω_∞, figure 1, (see appendix) the following reciprocity relation is easily derived,

$$\int_\Omega m^*_{ij}(s,q) w,_{ij}(q) d\Omega(q) = \int_\Omega m_{ij}(q) w^*_{ij}(s,q) d\Omega(q) \quad (7)$$

From equation (1), it is very easy to derive the integral representation of displacements. For a plate subjected to a transversal load g(s) applied over a particular region Ω_g, one can achieve,

$$C(S)w(S) + \int_\Gamma \left(V^*_n(S,Q)w(Q) - M^*_{nn}(S,Q)\frac{\partial w}{\partial n}(Q)\right) d\Gamma(Q) + \sum_{C=1}^{N_c} R^*_C(S,C) w_C(C) =$$

$$= \int_\Gamma \left(V_n(Q) w^*(S,Q) - M_{nn}(Q)\frac{\partial w^*}{\partial n}(S,Q)\right) d\Gamma(Q) + \sum_{C=1}^{N_c} R_C(C) w^*_C(S,C) +$$

$$+\int_{\Omega_g}\left(g(q)w^*\left(S,q\right)\right)d\Omega\left(q\right) \quad (8)$$

where S and Q are source and field points respectively, now taken along the boundary Γ.

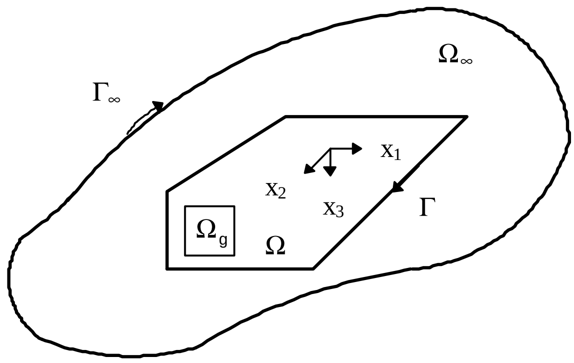

Figure 1. Plate domain inserted into the infinite space

In equation (8), w, $\partial w/\partial n$, M_{nn}, V_n and R_C are actual plate boundary values, deflections, rotations, normal bending moment, effective shear forces and corner reactions, respectively. The values indicated by * are the corresponding ones derived from the infinite domain problem with the source point defined at S. C(S) represents the well known free term that is the unit when S is an internal point, zero when outside defined and is equal to $\beta_C/2\pi$ for boundary points, being β_C the internal angle at S.

The above integral representation is enough to achieve a final set of algebraic equations. It is only required to define two collocation points to each discretization node and extra node corresponding to the geometrical corners (see ref. [5]). It is important to remember that one can find the second algebraic representation for each node using the slope integral equation which can be derived by differentiating equation (8). This scheme is the usual way adopted to solve plate bending problems by BEM (see refs. 1,2,3,4).

Let us now consider a more general plate bending problem characterised by exhibiting variable thickness, i.e. h = h(q). All relations presented in the previous section are still valid. The main difference to be introduced into the formulation appear in equation (7). One can derive the reciprocity relation for a field single point and then integrate it over the plate domain to obtain,

$$\int_{\Omega} m_{ij}(q)w^*_{ij}(s,q)d\Omega(q) = \int_{\Omega}\frac{D(s)}{D_0}m^*_{ij}(s,q)w,_{ij}(q)d\Omega(q) \quad (9)$$

where D_0 and D(q) are any reference and the field point stiffness values, respectively.

Now one has to work on equation (9) to transform it into the standard integral equation form. As a remaining domain term will remain in the final expression, it is necessary to decide which domain value or values will be appropriate as a problem unknown. Three kinds of domain integrals characterised by their densities and corresponding kernels are clearly possible.

The first possibility is given by successive integration by parts applied to all integral terms in equation (8) until eliminating all derivatives of the domain deflection values.

After the performing those integration accomplished by other appropriate changes the final integral representation is obtained, as follows,

$$
\begin{aligned}
C(S)w(S)\frac{D(S)}{D_0} &+ \frac{1}{D_0}\int_{\Gamma}\left(D(Q)V_n^*(S,Q) + 2\frac{\partial D(Q)}{\partial n}M_{ns}^*(S,Q) + \frac{\partial D(Q)}{\partial n}M_n^*(S,Q)\right)w(Q)d\Gamma(Q) \\
&- \frac{1}{D_0}\int_{\Gamma} D(Q)M_n^*(S,Q)\frac{\partial w}{\partial n}(Q)d\Gamma(Q) + \sum_{C=1}^{N_c}\frac{D_C}{D_0}R_C^*(S,C)w_C(C) = \\
&- \frac{1}{D_0}\int_{\Omega}\left(2\frac{\partial D(q)}{\partial x_i}q_i^*(S,q) + \frac{\partial^2 D(q)}{\partial x_i \partial x_j}m_{ij}^*(S,q)\right)w(q)d\Omega(q) \\
&= \int_{\Gamma}\left(V_n(Q)w^*(S,Q) - M_n(Q)\frac{\partial w^*}{\partial n}(S,Q)\right)d\Gamma(Q) + \sum_{C=1}^{N_c}R_C(C)w_C^*(S,C) + \\
&+ \int_{\Omega_g}\left(g(q)w^*(S,q)\right)d\Omega(q) \qquad (10)
\end{aligned}
$$

For the plate bending problems with varying thickness analysed in this work, the integral representation derived above is enough to give the necessary algebraic relations. Note that only deflections appear as internal values. Rotations and curvatures have been eliminated by performing integrations by parts.

In order to have a general BEM formulation of plate bending one can assume that expression (10) was derived for sub-regions were all required continuities are verified. Then, the whole domain is formed by joining all sub-regions. Some of the integration by parts performed to obtain equation (10) were not allowed without taking into account those discontinuity lines. By following this way one can realise that the only changes in equation (10) appear in its left hand side, due to abrupt changes in the thickness (interface between sub-regions of different thicknesses). Instead integrals over Γ, the general expression written from equation (10) must contain integral over $\Gamma + \Gamma_i$, Γ_i representing the interfaces. Moreover, internal corner reactions must be assumed at non smooth interface nodes (see ref. 9). As for this particular case of zoned problem there are two internal values (w and $\partial w / \partial n$), writing the slope equation is recommended, although selecting two collocations per each node to achieve the required number of algebraic relations is still possible.

The second possibility to transform equation (9) is given by choosing rotations as the variable to be left in the final expression. This alternative, obtained by proper transforming into equation (8), will be not treated in this work.

In the third situation we can work to leave the curvatures or the equivalent moment values inside the remaining domain integral. Assuming that the variation of the stiffness is given by $\Delta D(q) = D(s) - D_0$, equation (9) is transformed to

$$\int_{\Omega} m_{ij}(q) w^*_{ij}(s,q) d\Omega(q) = \int_{\Omega} m^*_{ij}(s,q) w,_{ij}(q) d\Omega(q) + \int_{\Omega} \frac{\Delta D(s)}{D_0} m^*_{ij}(s,q) w,_{ij}(q) d\Omega(q) \quad (11)$$

By integrating this equation by parts one can write the displacement integral representation for plate with varying stiffness or thickness. Although working with curvatures is possible (see ref. [8]) it is convenient replacing curvatures by moments leading to the well known integral representation of displacements with the presence of an initial moment field [6]. After carrying out the necessary changes, from equation (11) one can find,

$$C(S)w(S) + \int_{\Gamma} \left(V^*_n(S,Q) w(Q) - M^*_n(S,Q) \frac{\partial w}{\partial n}(Q) \right) d\Gamma(Q) + \sum_{C=1}^{N_c} R^*_C(S,C) w_C(C) =$$

$$= \int_{\Gamma} \left(V_n(Q) w^*(S,Q) - M_n(Q) \frac{\partial w^*}{\partial n}(S,Q) \right) d\Gamma(Q) + \sum_{C=1}^{N_c} R_C(C) w^*_C(S,C) +$$

$$+ \int_{\Omega_g} \left(g(q) w^*(S,q) \right) d\Omega(q) - \int_{\Omega} w^*_{,k\ell}(S,q) M^0_{k\ell}(q) d\Omega(q) \quad (12)$$

Equation (12) can also be adopted to analyse varying thickness plate bending problems. However, for that approach moment values must be taken as internal unknowns, requiring therefore three extra equations for each internal node defined inside the domain. In the next section a simple scheme to deal with varying stiffness plate bending problem, derived from the initial moment approach, will be discussed.

In order to solve plate problem integral representations of bending and twisting moments and shear forces are required. For the first described approach they are necessary to give the basic values of any analysis; if the third approach has being chosen they are required to define the main set of equations. For any case they are derived by differentiating conveniently equations (10) and (12).

From equation (10) moment and shear force integral representations for internal points (after making C(S)=1) can be derived by carrying out its second and third derivatives and applying the appropriate effort definitions given by equations (3). One must pay attention how to carry out the derivatives; For any case free terms will rise when performing derivatives of strong and hypersingular integral terms. By differentiating equation (10) twice and three times and introducing the definitions given by equations (3), one obtains,

$$m_{ij}(s) = -\int_{\Gamma}\left(D(Q)\overline{V}^{*}_{nij}(s,Q) + 2\frac{\partial D(Q)}{\partial s}\overline{M}^{*}_{nsij}(s,Q) + \frac{\partial D(Q)}{\partial n}\overline{M}^{*}_{nij}(s,Q)\right) w(Q)d\Gamma(Q)$$

$$-\int_{\Gamma} D(Q)\overline{M}^{*}_{nij}(s,Q)\frac{\partial w}{\partial n}(Q)d\Gamma(Q) - \sum_{C=1}^{N_c} D(C)\overline{R}^{*}_{Cij}(s,C)w_C(C)$$

$$+\int_{\Gamma}\left(V_n(Q)w^{*}_{ij}(s,Q) - M_n(Q)\frac{\partial w^{*}_{ij}}{\partial n}(s,Q)\right) d\Gamma(Q) + \sum_{C=1}^{N_c} R_C(C)w^{*}_{Cij}(s,C) +$$

$$+\int_{\Omega_g}\left(g(q)w^{*}_{ij}(s,q)\right) d\Omega(q) \quad (13)$$

Analogously, the shear force representation is given by,

$$q_i(s) = \int_{\Gamma}\left[D(Q)V^{*}_{ni}(s,Q) + 2\frac{\partial D(Q)}{\partial s}M^{*}_{nsi}(s,Q) + \frac{\partial D(Q)}{\partial n}M^{*}_{ni}(s,Q)\right] w(Q)d\Gamma(Q)$$

$$+\int_{\Gamma} D(Q)M^{*}_{nij}(s,Q)\frac{\partial w}{\partial n}(Q)d\Gamma(Q) + \sum_{C=1}^{N_c} D(C)R^{*}_{Cij}(s,C)w_C(C)$$

$$+\int_{\Gamma}\left(V_n(Q)w^{*}_{i}(s,Q) - M_n(Q)\frac{\partial w^{*}_{i}}{\partial n}(s,Q)\right) d\Gamma(Q) + \sum_{C=1}^{N_c} R_C(C)w^{*}_{Ck}(s,C) +$$

$$+\int_{\Omega_g}\left(g(q)w^{*}_{i}(s,q)\right) d\Omega(q) \quad (14)$$

Due to the order of singularities exhibited by the domain integrals, no free term arises when performing the required derivatives. Moreover, as the above expressions, equations (13) and (14), have been obtained for the particular case of continuous variation of stiffness and continuity of rotations, no domain integral last in the final representations of moments and shear forces (after the body discretization only collocations at body centre will be taken to compute moments and shear force values). Note that all kernels appearing in equations (13) and (14) are obtained by differentiating the corresponding values in equation (10) and applying the effort definitions, equations (3) and (4). For convenience the kernels derived for equations (13) and (14) are given in the appendix.

Following a similar way one can derive the corresponding integral representations for moments and shear forces for the initial moment approach (see reference 6),

$$m_{ij}(s) = -\int_{\Gamma}\left[V^{*}_{nij}(s,Q)w(Q) - M^{*}_{nij}(s,Q)\frac{\partial w}{\partial n}(Q)\right] - \sum_{c=1}^{N_c} R^{*}_{cij}(s,Q)w_c(Q) + \sum_{c=1}^{N_c} w^{*}_{cij}(s,Q)R_c(Q)$$

$$+\int_{\Gamma}\left[w^{*}_{nij}(s,Q)V_{n}(Q)-\frac{\partial w^{*}_{ij}}{\partial n}(s,Q)M_{n}(Q)\right]d\Gamma(Q)+\int_{\Omega_g} w^{*}_{ij}(s,q)g(q)d\Omega(q)$$

$$-\int_{\Omega} e^{*}_{,ijk\ell}(s,q)M^{p}_{k\ell}(q)d\Omega(q)-g_{ijk\ell}(q)M^{p}_{k\ell}(q)-M^{p}_{ij}(q) \quad (15)$$

Note that for this case a domain integral lasted in the representation, equation (15); this term exhibits singularity of order $1/r^2$, given by the kernel $e^{*}_{ijk\ell}(q,P)$, requiring therefore proper special care to be evaluated.

By applying the shear force definition, equation (4), one finds,

$$q_{\beta}(s)=-\int_{\Gamma}\left[V^{*}_{n\beta}(s,Q)w(Q)-M^{*}_{n\beta}(s,Q)\frac{\partial w}{\partial n}(Q)\right]d\Gamma(Q)$$

$$-\sum_{c=1}^{N_c}R^{*}_{c\beta}(s,Q)w_{c}(Q)+\sum_{c=1}^{N_c}w^{*}_{c\beta}(s,Q)R_{c}(Q)+\int_{\Gamma}\left[w^{*}_{\beta}(s,Q)V_{n}(Q)\right.$$

$$\left.-\frac{\partial w^{*}_{\beta}}{\partial n}(q,P)M_{n}(P)\right]d\Gamma(P)+\int_{\Omega_g}w^{*}_{\beta}(q,p)g(p)d\Omega(p)$$

$$-\int_{\Omega}e^{*}_{,\beta k\ell}(q,p)M^{p}_{k\ell}(p)d\Omega(p)-\overline{q}^{p}_{\beta}(q) \quad (16)$$

All the necessary kernels and the free term $g_{ijk\ell}(s)$, appearing in equations (15) and (16), are given in the appendix as well.

4 PLATE BENDING BEM ALGEBRAIC EQUATIONS

As usual for any BEM formulation, writing algebraic representations from the derived integral equations is required. For the present case, the plate boundary has been discretized into geometrically linear elements over which the boundary values have been approximated using quadratic shape functions. Discontinuities of the boundary values at the element ends are allow to be specified when convenient. The internal values are also approximated using linear shape functions defined over triangular cells.

The approximations of the boundary and internal values enable us to write algebraic representations of deflections, rotations, moments and shear forces starting from the integral forms derived in the last section. Bearing in mind that there are four boundary values at each node defined by the discretization, plus two extra values at each corner to take into account the corner reactions, we need to write an appropriate number of algebraic representations to have equal numbers of relations and unknowns to solve the problem with respect to boundary values. As already observed in previous studies [5], it is better to use only the displacement representation, writing those algebraic relations for appropriate boundary and external collocation points. For nodes where the continuity of the boundary values is assumed, we define two collocation points: one coincident with the node and another externally placed at a very small distance of the boundary. In order to deal with boundary value discontinuities, at corners for instance, it is convenient to define five collocation points associated with five unknowns (ten boundary values: w, $\partial w/\partial n$, V_n and M_n related to both element ends plus w_c and R_c, the corner values). Thus, four collocations are related with the element end unknowns, while the remaining collocation makes corner reaction and displacement independent values. Figure 2 illustrates a square plate discretization showing boundary and outside collocations, as well as the distance $d = a\ell_m$ between the out side points and the boundary, being ℓ_m the adjacent element average length and "a" a parameter defined within the range [0.01, 0.5].

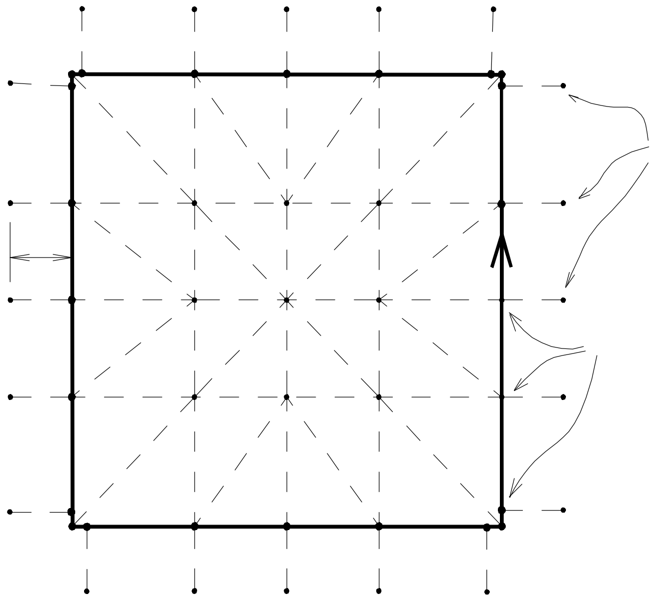

Figure 2. Boundary and internal discretizations. Collocation points.

Following the initial moment formulation given in Chueiri & Venturini[6], the algebraic system of equations, relating boundary values are obtained from equation (12) after approximating boundary and domain values as mentioned above, as follows:

$$\mathbf{HU} = \mathbf{GP} + \mathbf{T} + \mathbf{EM}^{\mathbf{o}} \quad (17)$$

where **U** and **P** are generalised boundary displacement and traction nodal values, **T** represents the loading and $\mathbf{M}^{\mathbf{o}}$ gives the initial moment values at internal and boundary nodes; **H** and **G** are the classical square matrices achieved by integrating all boundary elements, while the matrix **E** is obtained by performing the integrals over all cells.

Equation (15) can be algebraically written as well following the same steps:

$$\mathbf{M} = -\mathbf{H}'\mathbf{U} + \mathbf{G}'\mathbf{P} + \mathbf{T}' + (\mathbf{E}' - \mathbf{I})\mathbf{M}^{\mathbf{o}} \quad (18)$$

where **I** is the identity matrix.

The matrix equations (17) and (18) can be conveniently arranged to express the solution in terms of boundary values and moments $\mathbf{M}$, as follows (see ref. [6]):

$$\mathbf{X} = \mathbf{L} + \mathbf{R}\mathbf{M^o} \qquad \mathbf{M} = \mathbf{N} + \mathbf{S}\mathbf{M^o} \quad \text{(19a, b)}$$

In equations (19) **L** and **N** give the elastic solution, while the $\mathbf{M^o}$ effects are represented by the matrices **R** and **S**

Equations (19) can be adopted to solve non-linear problems, but can also be properly modified to consider regions of different stiffness. Let us consider a sub-region of a plate with stiffness D inside a domain characterized by the stiffness D_o. The moment value **M** can be obtained by correcting the value $\mathbf{M^e}$ (named here elastic moment and computed with D_o) by subtracting the initial moment $\mathbf{M^o}$, i.e. $\mathbf{M} = \mathbf{M^e} - \mathbf{M^o}$. Moreover $\mathbf{M^o}$ can be written in terms of **M** or $\mathbf{M^e}$, as indicated in figure 3. In particular we can write: $\mathbf{M^o}$ = $(D_o/D - 1)\mathbf{M}$. Thus, replacing this value into equation (19b) gives:

$$\mathbf{M} = (\mathbf{I} - \mathbf{S}\mathrm{T})^{-1}\mathbf{N} \quad (20)$$

where $T = (D_o/D - 1)$.

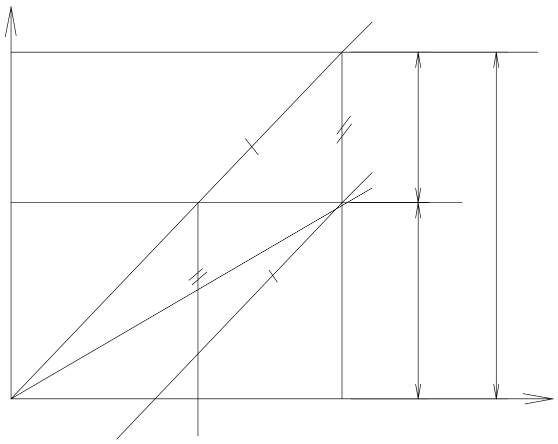

Figure 3. Moment relations for different stiffnesses.

Although this procedure is general and can be adopted to solve plates with any kind of thickness or stiffness variation, it requires the use of three internal values for each node, which makes very large the system of algebraic equations to be solved. For this reason we took the formulation that requires only the deflection value at each selected internal node.

Considering now the integral representation (10) for boundary and external collocations, the following set of algebraic equations can be write after assuming boundary values approximated along elements and internal values over cells:

$$[\mathbf{H_b} + \mathbf{S_{bb}}]\mathbf{U_b} + \mathbf{S_{bi}}\mathbf{U_i} = \mathbf{G_b}\,\mathbf{P_b} + \mathbf{T_b} \quad (21)$$

where the matrices **S** are obtained by integrating the domain term related with boundary ($\mathbf{U_b}$) and internal ($\mathbf{U_i}$) deflections; the subscripts b and i indicate boundary and internal values, **H** and **G** matrices are the usual ones for BEM formulations and T represents the domain load influences.

A similar matrix representation can also be written for internal nodes, that together with equation (21) gives,

$$\begin{bmatrix} \mathbf{H_b} + \mathbf{S_{bb}} & \mathbf{S_{bi}} \\ \mathbf{H_i} + \mathbf{S_{ib}} & \mathbf{I} + \mathbf{S_{ii}} \end{bmatrix} \begin{Bmatrix} \mathbf{U_b} \\ \mathbf{U_i} \end{Bmatrix} = \begin{bmatrix} \mathbf{G_b} \\ \mathbf{G_i} \end{bmatrix} \mathbf{P_b} + \begin{Bmatrix} \mathbf{T_b} \\ \mathbf{T_i} \end{Bmatrix} \quad (22)$$

where the second matrix line represents the algebraic relation for internal points and I stands for the unit matrix. Corner values are being considered together with the corresponding boundary values indicated by the subscript b.

Equation (22) is the algebraic representation of plates with varying thickness or stiffness. After solving this system ones has boundary values and internal deflections to used in the discretized forms of equations (13) and (14) to compute moments and shear forces.

5 NUMERICAL EXAMPLE

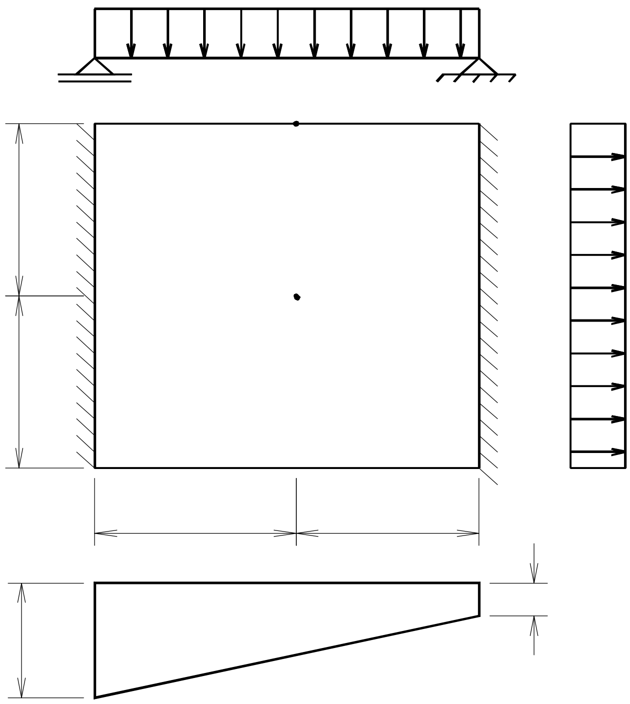

Figure 4. Plate definition

In order to illustrate the formulation presented in the forgoing sections a square plate exhibiting varying stiffness has been taken. Two opposite sides are assumed simply supported while the other two have no deflection or slope value prescribed. A uniform distributed load q has been considered action over the whole plate. The elastic modulus E is assumed while the Poisson ration ν = 0.3 is specified. The side length is taken equal to a. The stiffness is assumed to vary from D at the left side to D/2 at the right side. For simplicity no variation is assumed along the opposite direction assumed D (figure 4).

Note that the formulation that considers internal deflection values, represented algebraically by equation (22) was used to run this problem. That is the only implemented so far. For sure the formulation based on initial moments will be more computer time consuming due to the number of unknowns per each defined internal point.

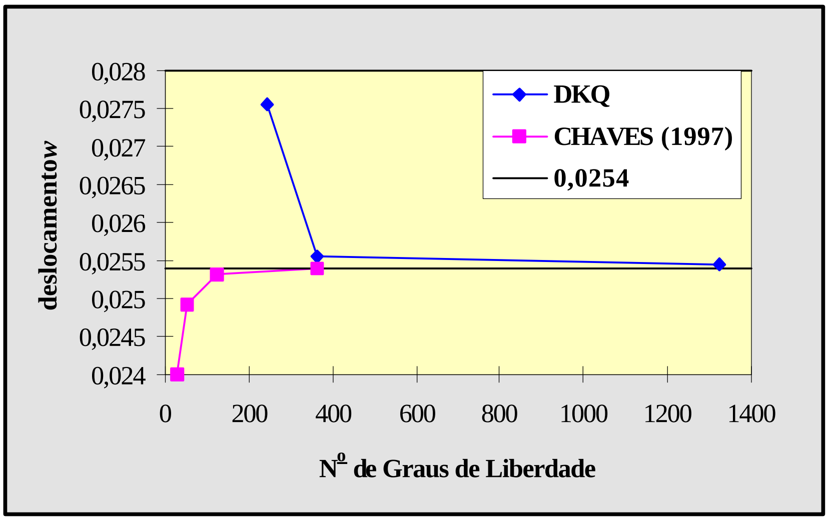

Figure 5. Displacements at the plate central point

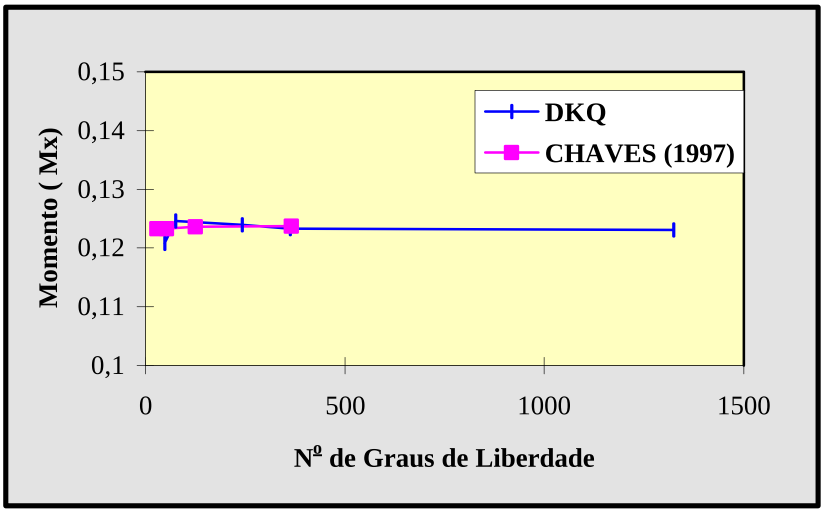

Figure 6. Moment M_x at the plate central point

Several discretizations have been adopted to run this problem, in particular we are shown the numerical responses computed using meshes of 4, 8, 16 and 32 boundary elements. In order to verify the accuracy of the BEM results, this problems was also numerically solved using a finite element code based on the DKT elements[11].

Figure 5 shows the deflection computed at the plate centre, comparing the results with those obtained with the FEM and the analytical solution[10]. Figures 6 and 7 illustrate the moment values ate the plate centre as well. All these results show that the BEM formulation are very accurate. Even using a rather coarse mesh one can achieve good solution in comparison with the finite element responses.

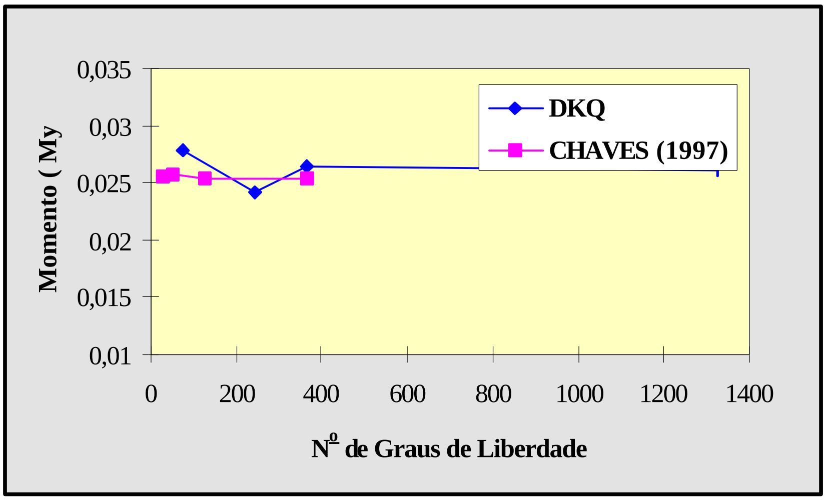

Figure 7. Moment M_y at the plate central point.

6 CONCLUSIONS

A BEM formulation to deal with Kirchhoff's plate bending problems exhibiting thickness or stiffness variable has been developed. Alternative ways to deal with this problem have been also discussed, basically showing what are possible transformations that can be performed in the domain integral to leave a convenient value at the internal points as degree of freedom . In particular, it has been successfully implemented a scheme that leave only unknown deflections at internal points. By the example taken to illustrate the formulation and by several other run the approach has shown to be very efficient and accurate. The accuracy is showing when comparing the obtained results with finite element or analytical solution. The efficiency is illustrated by the rather coarse discretization required to obtain relatively good results. Moreover, numerical solutions are not too sensitive to internal discretizations, which allow the use of domain coarse meshes with few internal points, therefore leading to small increases in the matrix size, without loosing accuracy.

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APPENDIX

The displacement fundamental value computed at the field point q due to an unit load applied at the source point, is expressed by,

$$w^*(s,q) = r^2(\ell nr - 1/2)/(8\pi D_o) \quad (A.1)$$

where r = r(s,q) is the distance between the points s and q, and D_o is a reference plate stiffness taken, for convenience, equal to the source point stiffness D(s).

By differentiating equation (A.1) and applying the appropriate definitions one can find the other required fundamental values:

$$\frac{\partial w^*}{\partial n} = \frac{r}{4.\pi.D_o}.\ln(r).\left(r,_i.n_i\right) \quad (A.2)$$

$$M_n^* = -\frac{1}{4.\pi}.\left[(1+\nu).\ln(r) + (1-\nu).\left(r,_i.n_i\right)^2 + \nu\right] \quad (A.3)$$

$$M_{ns}^* = -\frac{1}{4.\pi}.(1-\nu).\left(r,_i.n_i\right).\left(r,_j.s_j\right) \quad (A.4)$$

$$V_n^* = \frac{r,_i.n_i}{4.\pi.r}.\left[2.(1-\nu).\left(r,_j.s_j\right)^2 - 3 + \nu\right] + \frac{1-\nu}{4.\pi.R}.\left[1 - 2.\left(r,_i.s_i\right)^2\right] \quad (A.5)$$

$$R_C^* = M_{ns}^{*+} - M_{ns}^{*-} \quad (A.6)$$

where n and s represents the outward and tangent unit vectors, respectively, and the superscripts + and - are used to indicate values taken before and after the corner.

All kernels found when deriving moment and shear force representations, equations (11) and (12), are easily computed following the effort definitions, equations (3) and (4). For a symbolic fundamental value F^* the kernels $\overline{M}_{nij}^*, \overline{M}_{nsij}^*, \overline{V}_{nij}^*$ and $\overline{R}_{Cij}^*$ are given by the expression:

$$\overline{F}_{ij}^*(s,Q) = -.\left(\nu.\delta_{ij}.\frac{\partial^2 F^*}{\partial x_v.\partial x_v}(s,Q) + (1-\nu).\frac{\partial^2 F^*}{\partial x_i.\partial x_j}(s,Q)\right) \quad (A.7)$$

Similarly, the kernels appearing in equation (12) are given by,

$$\overline{F}_i^*(s,Q) = -\frac{\partial}{\partial x_i}\left(\cdot\frac{\partial^2 F^*}{\partial x_k \partial x_k}(s,Q)\right) \quad \text{(A.8)}$$

The remaining kernels in both equations w^*_{ij}, w^*_i, w^*_{Cij} and w^*_{Ci} are directly obtained from the fundamental value w^*, assuming the stiffness D equal the unit and applying formulas (A.7) and (A.8). Their final expressions can be conveniently derived the following operators:

$$w^*_{ij}(s,Q) = -.\left(\nu.\delta_{ij}.\frac{\partial^2 w^*}{\partial x_v.\partial x_v}(s,Q) + (1-\nu).\frac{\partial^2 w^*}{\partial x_i.\partial x_j}(s,Q)\right) \quad \text{(A.9)}$$

$$w^*_i(s,Q) = \frac{\partial}{\partial x_i}\left(\frac{\partial^2 w^*}{\partial x_k.\partial x_k}(s,Q)\right) = \frac{-r,_i}{2.\pi.r} \quad \text{(A.10)}$$

The kernels present in the initial moment integral representations, equations (13) and (14), can be easily derived using the second and third derivatives of the fundamental values $V^*_n(s,Q)$, $M^*_n(s,Q)$, $R^*_C(s,C)$, $w^*_C(s,C)$, $w^*(s,Q)$, $\partial w^*(s,Q)/\partial n$ and $w,^*_{k\ell}(s,q)$, applying the operator (A.9) and (A.10), now multiplied by the source point stiffness D(s).

The free term of equation (13), appeared due to the differentiation of a singular integral, is given by,

$$g_{ijk\ell}(s) = -\tfrac{1}{8}\left[(1-\nu)(\delta_{ik}\delta_{\ell j} + \delta_{kj}\delta_{i\ell}) + (1+3\nu)\delta_{ij}\delta_{k\ell}\right] \quad \text{(A.11)}$$