Economic Integration and Privatisation*

Juan Carlos Bárcena-Ruiz, María Begoña Garzón**

Departamento de Fundamentos del Análisis Económico, Universidad del País Vasco.
Avenida Lehendakari Aguirre, 83; 48015 Bilbao.

Abstract

In this paper we analyse whether it should be the government of each country that decides whether to privatise its public firms (non-integration) or whether this decision should be delegated to a supra-national authority (economic integration). We assume that there are two countries which form a single market in which there is free trade and each country has one public firm and \( n \) private firms. We show that if the supra-national authority decides whether or not to privatise public firms the aggregated political weighted welfare is no less than if the governments take this decision. We also show that the aggregated political weighted welfare is no less if these firms are owned by the governments instead of by the supra-national authority.

Key Words: international trade, economic integration, public firms, privatisation.

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** Departamento de Fundamentos del Análisis Económico, Universidad del País Vasco. Avenida Lehendakari Aguirre, 83; 48015 Bilbao, Spain. Phone: 34 + 94 601 38 15; Fax: 34 + 94 601 37 99. E-mail: jepgasab@bs.ehu.es (M. B. Garzón); jepbaruj@bs.ehu.es (J. C. Bárcena-Ruiz).
1. Introduction

In the late 1980’s there was a consensus in Western European countries in favour of a mixed economy including both public and private firms. Local and central governments in these countries intervened to create or purchase firms in a variety of sectors of the economy (including manufacturing and the service sector) for reasons which ranged from ideology to short-term strategic planning. As a result, EU governments own a significant percentage of the firms in the different sectors of industry in Europe.\(^1\)

In spite of this consensus, EU countries have privatised some of their public firms in recent years. In 1979 the United Kingdom privatised a large percentage of its public firms, and more privatisations, albeit on a smaller scale, followed in the rest of the then EC in the eighties. Subsequently, in the 1990’s the creation of the Single Market sparked further privatisation. However, in spite of all this privatisation no consensus was reached between EU countries regarding the percentage of firms in the different industries which should remain in public hands. Some governments favour going as far as possible with reorganisation and privatisation, while others prefer to continue with a policy of public ownership. At one extreme stands the UK, where most public firms have been privatised, and at the other stand Italy and Spain, where public ownership is still a major feature of many industries.

The Treaty of Rome is neutral as regards public ownership of firms. This neutrality is the result of the EU’s having to accommodate within it the different levels of state ownership of firms and the different opinions of its member countries. The Treaty does not question the ownership of public firms, but it does indicate that all public sector intervention must be neutral in the sense that it must not affect competition within the Single Market. For instance, there is a ban on state aid which might distort competition between member countries.

Parker (1998b, p. 23) argues that in spite of the neutral position maintained by the Treaty of Rome on public ownership, it has been recognised on some occasions that privatisation can

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\(^1\) For a more detailed analysis of this question see, for instance, Parker (1998a) and McGowan (1993).
be beneficial. The competitive environment created with the implementation of the Single Market has led Member States to take stock of the benefits they obtain by holding on to state ownership in some firms. For instance in 1994 the European Commission indicated that the privatisation of public firms could, when governments deemed it compatible with their objectives, help to improve the competitive environment (European Commission, 1994, p. 11). In regard to this last question it must be said that some countries are reluctant to privatise their state owned firms for fear that many will fall into foreign ownership.\(^2\) On the other hand, States are reluctant to transfer national sovereignty over public firms to Brussels. (Parker, 1998b, p. 22).

These arguments have led us to wonder from a theoretical viewpoint whether it is welfare-superior for each government to decide whether it privatises its public firms or for a supra-national body to be created which can decide whether to privatise public firms in different countries. For this latter case we also analyse whether ownership of these firms should be passed over to the supra-national authority.

The literature which looks at the decision whether to privatise public firms usually considers there to be only one country and one public firm (see for instance De Fraja & Delbono, 1989, 1990; White, 1996; Pal & White, 1998; Willner, 1999). This literature has been extended to consider privatisation decisions when there is international trade (see for instance Fjell & Pal, 1996; Pal & White, 1998), but the extensions continue to consider only one public firm, with firms selling their products in a single country. This literature does not analyse whether it should be the government of each country or a supra-national authority which decides whether public firms in the different countries should be privatised.

\(^2\) Parker (1998b, p. 37) indicates that EU countries fear that the privatisation of their public firms will lead to a loss of national control over major industries, and that this has led some countries to limit the percentage of privatised firms that can be acquired by foreign investors. For instance in Belgium sales have usually taken place on the basis of a preselection of investors rather than through public share offers. In Spain, the privatisation processes staged in the 1980’s involved the sale of minority share packages because the government did not wish to relinquish control of the firms involved.
To analyse this idea, we consider two countries which form a single market in which there is free trade. Each country has one public firm and $n$ private firms, which produce a homogeneous good with the same technology. If a public firm is owned by a government, its objective function is the political weighted welfare or political benefit of that country, which includes the consumer and producer surpluses of the country. If the public firms are owned by a supra-national authority their objective function is the aggregated political benefit of the countries. We consider three possibilities: the first is that each public firm is owned by one government, which decides whether to privatise it or not (non-integration). In the second case there is a supra-national authority which decides whether or not to privatise public firms, but these firms are owned by the governments (partial integration). In the third and final case there is a supra-national authority which owns the public firms and decides whether these firms are privatised (full integration).

For the sake of simplicity and in order to concentrate on whether or not the decision to privatise should be delegated to a supra-national authority, we do not here consider the income other than profits which public firms generate within their countries, which can affect the decision whether to privatise (in this regard, see Willner, 1999). Nor do we consider decisions made by politicians to achieve political objectives (Boycko et al., 1996, analyse this question), the existence of economies of scale (see Estrin & de Meza, 1995), differentiation in the goods produced by the firms (see Cremer et al., 1991), the fact that public firms are directed by managers (this question is analysed by Barros, 1995, Willner & Parker, 2000 and White, 2001), mergers between private and public firms (see Bárcena-Ruiz and Garzón, 2002), the regulation of public firms (see Bös, 1998) or the partial privatisation of those firms (see Matsumura, 1998).

Comparing the non-integration and partial integration cases, when the consumer and producer surpluses have the same weight in political benefit, we obtain that if the number of private firms is low enough no public firm is privatised in either case (since market competition would be greatly reduced). The aggregated political benefit of the two countries would therefore be the same in both cases. If the number of private firms is high enough, the aggregated political benefit is greater if countries integrate partially: under non-integration the governments decide whether to privatise for strategic reasons, but under partial integration
the supra-national authority decides whether or not to privatise for reasons of efficiency. As a result, the aggregated political benefit is greater in the latter case than in the former.

Under partial and full integration the decision whether to privatise is delegated to a supra-national authority. However in the former case the public firms are owned by the governments, while in the latter they are owned by the supra-national authority. A comparison of these two cases shows that if the number of private firms is low (high) enough, neither public firm is (both are) privatised, so the aggregated political benefit obtained is the same in both cases. When there is an intermediate number of private firms, we obtain that political benefit is greater under partial integration than under full integration. The intuition behind this result is that in the latter case the public firms are more aggressive (produce more) than in the former case (since their objective functions are the aggregated political benefit), and this reduces producers’ surpluses excessively even though it increases the surplus of consumers.

When the consumer surplus has a greater weight than the producer surplus in political benefit, we also obtain that the aggregated political benefit is no less under partial integration than under non-integration. Under partial integration political benefit is also no less than under full integration since, in this last case, public firms are excessively aggressive.

In this paper we also compare the marginal cost of the public firm with the market price. De Fraja and Delbono (1989) show, considering one public firm and one country, that the marginal cost of the public firm equals the market price. Fjell and Pal (1996), by considering that there are both domestic and foreign private firms, show that the marginal cost of the public firm exceeds the market price. We show in this paper that under non-integration the marginal cost of the public firm exceeds (equals) the market price if the consumer surplus has a greater (the same) weight than the producer surplus in political benefit. Under full integration, the marginal cost of the public firm is lower than the market price if and only if the weight of the consumer surplus in political benefit is low enough.

The paper is organised as follows. Section 2 presents the model. Section 3 shows the results obtained under non-integration if the political weights are equal. Section 4 shows the results obtained under integration if the political weights are equal. Section 5 compares the
results obtained in the preceding sections. Section 6 studies the case in which political weights differ and, finally, conclusions are drawn in section 7.

2. The Model

We consider a single market comprising two countries, $A$ and $B$. In each country there is one public firm, denoted by 0, and $n$ private firms ($n \geq 1$) and all firms produce a homogeneous good. For the sake of simplicity, we assume that there is free trade, there are no transportation costs and there is no possibility of discriminating between consumers from different countries. Therefore, consumers of both countries can buy the product from either a domestic or a foreign firm.

The public firm of each country can be privatised, and this decision can be taken by the governments or by a supra-national authority. Therefore, three cases arise. In the first case, each public firm is owned by a government which decides whether to privatisate it (non-integration). In the second case there is a supra-national authority which decides whether or not to privatisate public firms, but those firms are owned by the governments (partial integration). In these two cases we assume that if a public firm is privatised, it is acquired by domestic investors, so that, there are then $n+1$ private firms in that country.$^3$ In the third case, there is a supra-national authority which owns the public firms and decides whether those firms are privatised (full integration).

The inverse demand function for the product in country $k$ is: $p = a - 2y_k \ (k = A, B)$, where $p$ is the price for the good in the world market and $y_k$ is the amount of the good sold in country $k$. The world inverse demand function for the product is: $p = a - y_A - y_B$, where $y_A^++y_B^+ = q_0^+$, $\sum_{i=1}^{n} q_{Ai} + q_{B0} + \sum_{i=1}^{n} q_{Bi}$. Let $q_{ki}$ denote the amount of the good that the firm $i$ located

$^3$ We consider this assumption since, as pointed out in note 2, EU countries fear that the privatisation of their public firms will lead to a loss of national control over major industries. It can be shown that the main results of the paper hold if we permit public firms to be acquired by foreign investors.
in country $k$, firm $ki$, sells in the single market ($k=A, B; i=0, 1, \ldots, n$). The consumer surplus in country $k$, denoted by $CS_k$, is: $CS_k = (y_k)^2, k=A, B$.

As is usually assumed in literature on mixed oligopoly (see, for example, De Fraja and Delbono, 1989, 1990), firms have the same technology, represented by the following cost function: $C(q_{ki}) = (q_{ki})^2, k=A, B; i=0, \ldots, n$. The profit function of firm $ki$ is:

$$\pi_{ki} = (a-q_{A0} - \sum_{j=1}^{n} q_{Aj} - q_{B0} - \sum_{j=1}^{n} q_{Bj})q_{ki} - q_{ki}^2, i=0, \ldots, n; k=A, B.$$  \hspace{2cm} (1)

We consider governments as seeking to maximize political weighted welfare or political benefit. The objective function considered by government $k$ includes the consumer surplus in country $k$, $CS_k$, and the producer surplus in country $k$, $PS_k$. As usual, the producer surplus in country $k$ is $PS_k = \sum_{i=0}^{n} \pi_{ki}$. We use the following expression to measure political benefit:

$$W_k = \alpha CS_k + PS_k, 1 \leq \alpha < 2, k=A, B.$$  \hspace{2cm} (2)

We propose a three stage game with the following timing. In the first stage, we analyse whether the decision whether to privatise public firms should be taken by governments or by a supra-national authority. In the second stage, under non-integration, the two governments decide simultaneously whether they privatise their public firms or not; under economic integration, the supra-national authority decides whether to privatise the two public firms,

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4 It can be shown that if the cost function is given by $C(q_{ki}) = \frac{c}{2} (q_{ki})^2$, the main results of the paper hold; i.e. the results of the paper are robust to changes in parameter $c$.

5 We assume that $\alpha<2$ to assure that public firms produce a positive output level. We consider that $\alpha\geq1$ to assure that the consumer surplus has a weight which is no less than the producer surplus in political benefit (this assumption is also made by Matsumura, 1998). If $\alpha<1$ the results obtained in the paper can change and “non-realistic” equilibria can arise, since the “excessively low” weight of the consumer surplus distorts the objective function of public firms.
only one of them or neither of them. In the third stage, each firm decides its output level. We solve the game by backward induction to obtain a subgame perfect Nash equilibrium.

3. Non-integration

In this section we assume that each government decides whether to privatise its public firm or not. Given that there are two public firms that can be privatised, one in each country, there are four subgames that, by symmetry, can be reduced to three. These three subgames are the following: neither of the governments privatises (denoted by superscript \(NN\)), each government privatises (denoted by superscript \(PP\)), and one government does not privatise while the other government does; in this last case, we use the superscript \(NP\) to denote the country that does not privatise and the superscript \(PN\) to denote the other. Next we analyse each of the different subgames.

3.1. Neither government privatises

In the third stage of the game, the private firm \(ki\) chooses the output level, \(q_{ki}\), that maximizes its profit function; the public firm \(0\) owned by government \(k\) chooses the output level, \(q_{k0}\), that maximizes its political benefit. We get the following result.

**Lemma 1.** Under non-integration, when neither government privatises, the output level of firms, the consumer surplus, the profit of firms, the producer surplus and the political benefit in each country are, respectively:

\[
q_0^{NN} = \frac{a(3-n(1-\alpha))}{15+4n-3\alpha}, \quad q_i^{NN} = \frac{a(3-\alpha)}{15+4n-3\alpha}, \quad CS^{NN} = \frac{a^2(3+2n)^2}{(15+4n-3\alpha)^2},
\]

\[
\pi_i^{NN} = \frac{2a^2(3-\alpha)^2}{(15+4n-3\alpha)^2}, \quad \pi_0^{NN} = \frac{a^2(3-n+n\alpha)(6+n-3\alpha-n\alpha)}{(15+4n-3\alpha)^2},
\]

\[
PS^{NN} = \frac{a^2(9(2-\alpha)-n^2(1-\alpha)^2+n(15-6\alpha-\alpha^2))}{(15+4n-3\alpha)^2},
\]

\[
W_i^{NN} = \frac{a^2(18+n(15+6\alpha-\alpha^2)-n^2(1-6\alpha+\alpha^2))}{(15+4n-3\alpha)^2}, \quad i=1, \ldots, n.
\]
Using the output levels obtained in lemma 1 we can compare the market price and the marginal cost of the public firm. We show the result in the following lemma.

**Lemma 2.** *Under non-integration, when neither government privatises, the marginal cost of the public firm equals the market price if and only if \( \alpha = 1 \). The marginal cost of the public firm exceeds the market price if \( 2 > \alpha > 1 \).*

This lemma shows that the marginal cost of the public firm equals the market price if the consumer and producer surpluses have the same weight in political benefit (\( \alpha = 1 \)). If \( \alpha = 1 \), each public firm chooses the output level that maximises the political benefit of its government, taking into account that the consumer surplus has the same weight as the producer surplus in political benefit. Therefore, given the symmetry existing in this case, we obtain the same result than DeFraja and Delbono (1989) although we consider that there are two public firms instead of one.

When the political weights differ, we obtain that the marginal cost of the public firm exceeds the market price. This result is also obtained by Fjell and Pal (1996) considering an industry with one public firm, \( m \) domestic private firms and \( n \) foreign private firms. In their model, for a given amount of total private firms’ output, the output level of the public firm is greater if some output comes from foreign firms, since foreign firms’ profit is not included in political benefit; therefore the producer (consumer) surplus has a lower (greater) weight in political benefit. As a result, they obtain that the public firm’s marginal cost equals the price only if \( n = 0 \). In our paper, when \( \alpha \) increases, the output level of the public firms increases since the weight of the consumer surplus is greater; as a result, the marginal cost of the public firms exceeds the price.

### 3.2. Only one government privatises

We assume in this case that only one government privatises. For example, suppose that firm 0 in country \( A \) is sold to domestic investors. At the third stage, each of the \( 2n+1 \) private firms chooses the profit-maximising output level, while \( q_{B0} \) maximises government \( B \)’s political benefit function. We get the following result.
Lemma 3. Under non-integration, when only one government privatises, the output level of firms, the consumer surplus, the profit of firms, the producer surplus and the political benefit in each country are, respectively:

\[ q_0^{NP} = \frac{\alpha(6 - 2n(1 - \alpha) + \alpha)}{30 + 10n - 3\alpha}, \quad q_i^{PN} = q_j^{PN} = q_j^{NP} = q_j^{PN} = \frac{a(6 - \alpha)}{30 + 10n - 3\alpha}, \quad CS^{NP} = CS^{PN} = \frac{a^2(6 + 5n)^2}{(30 + 10n - 3\alpha)^2}, \]

\[ \pi_0^{NP} = \frac{2a^2(6 + n - 2\alpha - n\alpha)(6 - 2n + \alpha + 2n\alpha)}{(30 + 10n - 3\alpha)^2}, \quad \pi_i^{NP} = \pi_j^{PN} = \frac{2a^2(6 - \alpha)^2}{(30 + 10n - 3\alpha)^2}, \]

\[ PS^{NP} = \frac{2a^2(2(18 - 3\alpha - \alpha^2)) + n(30 - \alpha - 4\alpha^2) - 2n^2(1 - \alpha)^2}{(30 + 10n - 3\alpha)^2}, \quad PS^{PN} = \frac{2a^2(1 + n)(6 - \alpha)^2}{(30 + 10n - 3\alpha)^2}, \]

\[ W^{NP} = \frac{a^2(4(18 + 6\alpha - \alpha^2) + n(60 + 58\alpha - 8\alpha^2) - n^2(4 - 33\alpha + 4\alpha^2))}{(30 + 10n - 3\alpha)^2}, \quad W^{PN} = \frac{a^2(2(36 + 6\alpha + \alpha^2) + 2n(36 + 18\alpha + \alpha^2) + 25n^2\alpha)}{(30 + 10n - 3\alpha)^2}, \quad i = 1, \ldots, n; \quad j = 0, \ldots, n. \] (4)

If one public firm is privatised it reduces its output level \( q_0^{PN} < q_0^{NN} \) since its objective function is now its profit. But this reduction in output is not compensated by a sufficient variation in the output level of the other public firm \( q_0^{PN} + q_0^{NP} < 2q_0^{NN} \), since the objective function of the latter comprises only the political benefit of its country. As a result, the industry output level decreases after the privatisation of a public firm. In this case, as market competition decreases following privatisation, the profit of the public firm increases more than the profit of the private firms \( \pi_0^{NP} > \pi_i^{NP} = \pi_j^{PN} \) since, although its total costs are greater, its market share is also greater.

Lemma 4. Under non-integration, when only one government privatises, the marginal cost of the public firm is lower than the market price if and only if \( \alpha < \alpha^* \), where \( \alpha^* = \frac{6 + 4n}{5 + 4n}, \quad 2 > \alpha^* > 1. \)

This lemma shows that the market price is greater than the marginal cost of the public firm if the parameter \( \alpha \) is low enough \( (\alpha < \alpha^*) \). This is due to the fact that when one public firm is privatised, that firm reduces its production, which is not compensated by a sufficient variation in the output level of the other public firm, so the output of industry decreases after
privatisation. If \( \alpha \) is high enough \((\alpha \geq \alpha^*)\), as the output level of the public firm increases with \( \alpha \), the reduction in the output level of the privatised firm is compensated by a sufficient variation in the output level of the other public firm. Therefore, in this last case the marginal cost of the public firm exceeds the price.

3.3. The two governments privatise

In this case, each government privatises its public firm, which is acquired by domestic investors. Therefore, in each country there are \( n+1 \) private firms. Each private firm chooses the output level that maximizes its profit function. We get the following result.

**Lemma 5.** Under non-integration, when the two governments privatise, the output level of firms, the consumer surplus, the profit of firms, the producer surplus and the political benefit in each country are, respectively:

\[
q_{i}^{PP} = \frac{a}{5 + 2n}, \quad CS^{PP} = \frac{a^2(1 + n)^2}{(5 + 2n)^2}, \quad \pi_{i}^{PP} = \frac{2a^2}{(5 + 2n)^2},
\]

\[
PS^{PP} = \frac{2a^2(n + 1)}{(5 + 2n)^2}, \quad W^{PP} = \frac{a^2(n + 1)(2 + \alpha + n\alpha)}{(5 + 2n)^2}, \quad i = 0, \ldots, n.
\]

Once we have solved the different subgames, we have to solve stage two. We analyse first the case in which the consumer and the producer surpluses have the same weight in political benefit \((\alpha = 1)\). Next, in section 6, we shall study whether the results are robust to changes in parameter \( \alpha \).

3.4. The decision by the governments whether to privatise

By using lemmas 1, 3 and 5 we can compare the consumer and producer surpluses obtained in the different subgames. This comparison is useful to study the decision by governments whether to privatise their public firms.

**Lemma 6.** Under non-integration, \( CS^{NN} > CS^{NP} = CS^{PN} > CS^{PP} \) and \( PS^{PP} > PS^{PN} > PS^{NN} \).
Given that each public firm chooses the output level that maximizes the political benefit of its government, it is more aggressive in the product market than private firms (i.e. its output level is greater). As a result, the highest industry output level is obtained when there are two public firms in the market and the lowest industry output level is obtained when all firms are private. Therefore, as the consumer surplus increases with industry output level, we get that $CS^NN > CS^NP = CS^PN > CS^{PP}$.

The lowest (greatest) producer surplus is obtained when there are two (no) public firms in the market since the greatest (lowest) industry output level is obtained in that case. When only one government privatises, the producer surplus takes an intermediate value. As we have seen in lemma 3, $q_0^{NN} > q_0^{PN}$ and $2q_0^{NN} > q_0^{PN} + q_0^{NP}$. As a result, the industry output level decreases after the privatisation of one public firm. Thus, the producer surplus is lower when there are two public firms than when there is only one and, in this last case, the producer surplus is lower than when all firms are private. When there is only one public firm in the market, the producer surplus is greater in the country that owns that firm ($PS^NP > PS^{PN}$) since, as we have seen in lemma 3, the public firm makes a greater profit than the private firms ($\pi_0^{NP} > \pi_j^{NP} = \pi_j^{PN}$).

Next we solve the second stage of the game under non-integration. The following result is obtained.

**Proposition 1.** Under non-integration, if $n > n_\alpha$ only one government privatises while if $n \leq n_\alpha$ neither government privatises, where $n_\alpha = 13.0316$.

**Proof.** From lemmas 1, 3 and 5 we obtain that $W^{NN} - W^{PN} = a^2(369 + 102n - 10n^2) / 8(3 + n)^2(27 + 10n)^2$, which is positive if and only if $n < n_\alpha = 13.0316$. Therefore, if one government does not privatises, the other government privatises (does not privatises) if $n > n_\alpha$ ($n \leq n_\alpha$). On the other hand, $W^{NP} - W^{PP} = a^2(113 + 54n + 4n^2) / (5 + 2n)^2(27 + 10n)^2$, which is positive for all $n$. Thus, if one government privatises the other government does not privatises. \[\square\]
We consider first the decision taken by one government when the other government does not privatise. When the parameter $n$ is great enough ($n > n_\alpha$), if one government does not privatise the other government privatises ($W^{PN} > W^{NN}$). In this case, the producer surplus has a greater weight than the consumer surplus in political benefit since the competition in the product market is high enough (and $CS^{NN} > CS^{PN}$ while $PS^{NN} < PS^{PN}$). When $n$ is low enough ($n \leq n_\alpha$), if one government does not privatise nor does the other ($W^{NN} > W^{PN}$). In this case, the consumer surplus has a greater weight than the producer surplus in political benefit since the competition in the product market is low enough.

The decision taken by one government when the other government privatises remains to be analysed. In this case, independently of $n$, when one government privatises, the other government does not privatise ($W^{NP} > W^{PP}$). As we have seen in lemma 6, when there is only one public firm in the market the strategic behaviour of the public firm causes the producer surplus to be greater in the country that owns that firm ($PS^{NP} > PS^{PP}$). This implies that, although $PS^{NP}$ is lower than $PS^{PP}$, the difference between $PS^{NP}$ and $PS^{PP}$ is small. On the other hand, $CS^{NP}$ is greater than $CS^{PP}$ and the difference between $CS^{NP}$ and $CS^{PP}$ is great since the strategic behaviour of the public firm affects the consumer surplus of both countries equally. Thus, the difference between $CS^{NP}$ and $CS^{PP}$ is greater than the difference between $PS^{PP}$ and $PS^{NP}$.

As a result of the preceding arguments we obtain that when $n$ is low enough ($n \leq n_\alpha$), neither government privatises its public firm. When $n$ is high enough ($n > n_\alpha$) only one government privatises, and therefore there are two equilibria; in this case, the government which does not privatise gets greater political benefit than the other government, but both governments are better off than if neither public firm is privatised. It must be noted that, in this case, firms are never both privatised.

The result obtained in this proposition is different from that obtained by De Fraja and Delbono (1989). They consider only one country and one public firm, so they cannot obtain the result in which only one public firm is privatised since this result is driven by strategic reasons.
4. Economic Integration

In this section we assume that there is a supra-national authority which decides whether or not to privatise the two public firms. We consider two types of economic integration: partial integration and full integration.

4.1. Partial integration

Under partial integration there is a supra-national authority which decides whether the public firms are privatised, but these firms are owned by governments. In this case, the following result is obtained.

**Proposition 2.** Under partial integration, if \( n > n_\gamma \) the supra-national authority privatises both public firms, if \( n_\gamma \geq n > n_\beta \) the supra-national authority privatises only one and if \( n \leq n_\beta \) neither of them is privatised, where \( n_\beta = 1.5917 \) and \( n_\gamma = 1.8364 \).

**Proof.** From lemmas 1, 3 and 5 we obtain that

\[
W^{NP} + W^{PN} - 2W^{NN} = \frac{a^2(-153 + 42n + 34n^2)}{4(3+n)^2(27+10n)^2},
\]

which is positive if and only if \( n > n_\beta = 1.5917 \); \( 2W^{PP} - W^{AP} + W^{PN} = \frac{4a^2(-19 + 3n + 4n^2)}{(5+2n)^2(27+10n)^2} \), which is positive if and only if \( n > n_\gamma = 1.8364 \); \( 2W^{PP} - 2W^{NN} = \frac{a^2(-9 + 2n + 2n^2)}{4(3+n)^2(5+2n)^2} \), which is positive if and only if \( n > n_\delta = 1.6794 \). □

The intuition underlying this result is the following. When the number of private firms is low enough \( (n \leq n_\beta) \), as the competition in the product market is also low enough, the privatisation of either of the public firms causes a reduction in the consumer surplus that is greater than the increase in the producer surplus. Thus, in this case neither public firm is privatised by the supra-national authority. When \( n \) is great enough \( (n > n_\gamma) \), the competition in the product market is also great enough and thus both public firms are privatised since the
reduction in the consumer surplus is compensated by an increase in the producer surplus. As a result, in this case the supra-national authority privatises both public firms.

When the number of private firms takes an intermediate value \( (n_{\gamma} \geq n > n_{\beta}) \) the supranational authority privatises only one public firm. Under partial integration, although it is the supra-national authority that decides whether to privatise public firms, these firms are owned by governments; thus, each public firm maximizes the political benefit of its country. This means, as we have seen in lemma 3, that if one public firm is privatised the reduction in its output level is not compensated by a sufficient increase in the output level of the other public firm (since its objective function comprises only the political benefit of its country).\(^6\) In this interval, although the consumer surplus decreases after the privatisation of one public firm, this reduction is compensated by an increase in the producer surplus (since, as we have seen in lemma 3, the profit of the public firm rises). This means that when the number of private firms takes an intermediate value only one public firm is privatised by the supranational authority. It must be noted that, as the two countries are identical, either of the two public firms could be privatised.

4.2. Full integration

Under full integration there is a supranational authority which owns the public firms and decides whether they are privatised. If neither public firm is privatised, the two public firms choose the output level that maximises the aggregated political benefit of the two countries. In this case, we obtain the following result.

\(^6\) In fact, under partial integration, \( q^N_{o} > q^N_{o} \) if and only if \( n < 1.5 \). If the objective function of the public firm is the aggregated political benefit of the two countries, when one public firm is privatised the other public firm increases its output level (as much as possible) to compensate the reduction in the output level of the privatised firm.
Lemma 7. Under full integration, when neither government privatises, the output level of the firms, the consumer surplus, the profit of the firms, the producer surplus and political benefit in each country are, respectively:

\[ q_0^{NN} = \frac{a(3 - 2n(1 - \alpha))}{18 + 4n - 6\alpha}, \quad q_i^{NN} = \frac{a(2 - \alpha)}{9 + 2n - 3\alpha}, \quad CS^{NN} = \frac{a^2(3 + 2n)^2}{4(9 + 2n - 3\alpha)^2}, \]

\[ \pi_i^{NN} = \frac{2a^2(2 - \alpha)^2}{(9 + 2n - 3\alpha)^2}, \quad \pi_0^{NN} = \frac{a^2(27 - 18\alpha - 4n(n + 3)(1 - \alpha)^2)}{4(9 + 2n - 3\alpha)^2}, \]

\[ PS^{NN} = \frac{a^2(27 - 18\alpha + 4n(5 - 2\alpha - \alpha^2) - 4n^2(1 - \alpha)^2)}{4(9 + 2n - 3\alpha)^2}, \]

\[ W^{NN} = \frac{a^2(9(3 - \alpha) + 4n(5 + \alpha - \alpha^2) - 4n^2(1 - 3\alpha + \alpha^2))}{2(9 + 2n - 3\alpha)^2}, \quad i = 1, \ldots, n. \quad (6) \]

If both public firms are privatised we obtain the same result as in lemma 5, since all firms are private. Finally, if there is only one public firm, it chooses the output level that maximises the aggregated political benefit of the two countries. In this case we obtain the following result.\(^7\)

Lemma 8. Under full integration, when only one government privatises, the output level of firms, the consumer surplus in each country, the profit of firms, the aggregated producer surplus and the aggregated political benefit are, respectively:

\[ q_0^{NP} = \frac{a(2 + \alpha - 2n(1 - \alpha))}{14 + 4n - 3\alpha}, \quad q_i^{NP} = q_j^{NP} = \frac{a(3 - \alpha)}{14 + 4n - 3\alpha}, \]

\[ CS^{NP} = CS^{PN} = \frac{a^2(5 + 4n)^2}{4(14 + 4n - 3\alpha)^2}, \quad \pi_0^{NP} = \frac{a^2(2 - 2n + \alpha + 2n\alpha)(7 + 2n - 4\alpha - 2n\alpha)}{(24 + 4n - 3\alpha)^2}, \]

\[ \pi_i^{NP} = \pi_j^{PN} = \frac{2a^2(3 - \alpha)^2}{(14 + 4n - 3\alpha)^2}, \quad i = 1, \ldots, n; \quad j = 0, \ldots, n, \]

\[ PS^{NP} + PS^{PN} = \frac{a^2(32 - 13\alpha - 2\alpha^2 + n(26 - 4\alpha - 6\alpha^2) - 4n^2(1 - \alpha)^2)}{(14 + 4n - 3\alpha)^2}, \]

\[ W^{NP} + W^{PN} = \frac{a^2(64 - \alpha - 4\alpha^2 + 4n(13 + 8\alpha - 3\alpha^2) - 8n^2(1 - 4\alpha + \alpha^2))}{2(14 + 4n - 3\alpha)^2}. \quad (7) \]

\(^7\) It must be noted that when \(\alpha = 1\), we consider the same model as De Fraja and Delbono (1989), since they assume only one country and one public firm.
We analyse first the case in which the consumer and the producer surpluses have the same weight in political benefit (\(\alpha = 1\)). In section 6 we shall study whether the results are robust to changes in parameter \(\alpha\).

**Proposition 3.** Under full integration, if \(n > n_\delta\) the supra-national authority privatises both public firms while if \(n \leq n_\delta\) neither of them is privatised, where \(n_\delta = 1.6794\).

**Proof.** From lemmas 5, 7 and 8 we obtain that 
\[W_{NP} + W_{PN} - 2W_{NN} = \frac{a^2(-27 + 2n + 4n^2)}{4(3 + n)^2(11 + 4n)^2},\]
which is positive if and only if \(n > n_\phi = 2.3600\); 
\[2W_{PP} - W_{NP} + W_{PN} = \frac{a^2(-23 + 12n + 8n^2)}{2(5 + 2n)^2(11 + 4n)^2},\]
which is positive if and only if \(n > n_\theta = 1.1040\); 
\[2W_{PP} - 2W_{NN} = \frac{a^2(-9 + 2n + 2n^2)}{4(3 + n)^2(5 + 2n)^2},\]
which is positive if and only if \(n > n_\delta = 1.6794\).

The difference between partial and full integration is that in the latter case public firms choose the output level that maximises the aggregated political benefit of the two countries while in the former case each public firm chooses the output level that maximizes the political benefit of its country. Thus, when only one public firm is privatised, under full integration the other public firm increases its output level (as much as possible) to compensate the reduction in the output level of the privatised firm; under partial integration the other public firm behaves strategically and, thus, its output level is lower than under full integration.

If the supra-national authority privatises only one public firm, the privatised firm reduces its output level, while the other public firm increases its output level. However, although the objective function of the public firm is the aggregated political benefit, this last firm cannot increase its output level enough to compensate the reduction in the output of the industry since its production costs are quadratic. Moreover, given that the production costs are quadratic, it is cheaper to produce a given output level using two public firms than using one. Therefore, the supra-national authority never privatises only one public firm. As a result of
the preceding arguments, when parameter \( n \) is low enough \((n \leq n_\delta)\), neither public firm is privatised. However, if \( n > n_\delta \), the supra-national authority privatises both of them since competition in the product market is great enough.

5. Comparison between the three types of integration when political weights are equal

In this section we solve the first stage of the game. Comparing the aggregated political benefit in the three types of integration we obtain the following results.

**Proposition 4.** When political weights are equal, the aggregated political benefit under partial integration is no less than under non-integration.

This proposition shows that non-integration is (weakly) dominated by partial integration. By comparing propositions 1 and 2 we get that if \( n \leq n_\beta \) the aggregated political benefit is the same under partial integration as under non-integration since in both cases neither public firm is privatised. When \( n > n_\beta \) under non-integration the governments decide, for strategic reasons, whether their public firms are privatised or not. However, under partial integration there is a supra-national authority which decides, for reasons of efficiency, whether the public firms are privatised. As a result, if \( n > n_\beta \) the aggregated political benefit is greater under partial integration than under non-integration, and the political benefit of each country could increase if the decision of privatisation were delegated to a supra-national authority.

If \( n_\gamma \geq n > n_\beta \), under partial integration only one public firm is privatised while under non-integration neither of them is privatised, and the aggregated political benefit is greater in the first case. In this interval, under non-integration, given that the country that privatises obtains a lower political benefit than the other country \((W_{NN} > W_{PN})\), neither of the two governments wants to privatise. Under partial integration, given that \( n \) takes an intermediate value, the supra-national authority privatises only one public firm. As in this interval \( W_{NP} > W_{NN} > W_{PN} \), the country that privatises would prefer not to delegate the decision whether to privatise to a supra-national authority (this country would prefer not to privatise). But, as \( W_{NP} + W_{PN} > 2W_{NN} \), the country that privatises could be compensated by the other country by way of a monetary

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8 It can be shown that, under full integration, \( q_0^{NP} > q_0^{NN} \) for all \( n \).
transference, and both countries would obtain a greater political benefit under partial integration than under non-integration.

If \( n_{δ} \geq n > n_{ρ} \) under partial integration both public firms are privatised while under non-integration neither of them is privatised, and the aggregated political benefit is greater in the first case. In this interval, as \( W^{NN} < W^{PP} \), the two governments obtain a greater political profit by delegating the decision whether to privatise to a supra-national authority.

Finally, when \( n > n_{δ} \), under partial integration both public firms are privatised while under non-integration only one is, and the aggregated political benefit is greater in the first case. In this interval \( W^{NP} > W^{PP} \) and, thus, the government which owns the public firm would not want to privatise it. But, as \( W^{NP} + W^{PN} > 2W^{PP} \), this government could be compensated by the other government by way of a monetary transference, and both countries would obtain a greater political benefit under partial integration than under non-integration.

Now that we have compared non-integration and partial integration, let us compare partial and full integration.

**Proposition 5.** When political weights are equal, the aggregated political benefit under partial integration is no less than under full integration.\(^9\)

This proposition shows that full integration is (weakly) dominated by partial integration. By comparing propositions 2 and 3 we get that if \( n \leq n_{β} \) the aggregated political benefit is the same in both cases since in neither case is either public firm privatised. Given that the consumer surplus has the same weight as the producer surplus in political benefit and that

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\(^9\) If we allow partial privatisation of public firms (see Matsumura, 1998; Bárccena-Ruiz and Garzón, 2002), the supra-national authority would choose the percentage of the shares of the public firms that is held by governments (under partial integration) or by the supranational authority (under full integration). In both cases, the supra-national authority chooses partially to privatise public firms (and, thus, we have semi-public firms). There is a greater aggregated political benefit under full integration than under partial integration since, in the first case, the supra-national authority can choose the level of “aggressiveness” of the public firms by choosing the percentage of the shares of the public firms that it wants to hold (the lower this percentage is, the lower the output level of the semi-public firms).
there are no asymmetries in the model, the output levels of both the public firms and the private firms (and, thus, the consumer and producer surpluses) are the same under partial and full integration. If $n \geq n_\gamma$, both public firms are privatised in both cases and, thus, the aggregated political benefit is also the same.

It only remains to compare the two types of integration in the interval $n_\gamma \geq n > n_\beta$. In this interval, as proposition 2 shows, under partial integration only one public firm is privatised since the aggregated political benefit is greater than if the two public firms are privatised or if neither of them is privatised. In this interval, under full integration, the supra-national authority privatises both public firms (if $n_\gamma \geq n > n_\delta$) or neither of them (if $n_\delta \geq n > n_\beta$). And it must be noted that when there are two public firms in the market, the aggregated political benefit is the same in the two types of economic integration; if all firms are private, the same aggregated political benefit is also obtained in both cases. Therefore, in this interval, partial integration generates a greater aggregated political benefit than full integration.

In the interval $n_\gamma \geq n > n_\beta$, under full integration neither public firm is privatised while under partial integration only one public firm is privatised. It must be noted that if neither public firm is privatised the same aggregated political benefit is obtained under full and partial integration. In this last type of integration, when only one government privatises the aggregated political benefit is greater than if neither government privatises ($W_{NP} + W_{PN} > 2W_{NN}$). However, when only one government privatises, this government obtains a lower political benefit than if neither government privatises ($W_{PN} < W_{NN}$). But, as $W_{NP} + W_{PN} > 2W_{NN}$, the government that privatises could be compensated by the other

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10 We show in section 6 that, when the consumer surplus has a different weight than the producer surplus in political benefit, the aggregate political benefit is different under partial and full integration.

11 An alternative explanation for this result is the following. In the interval $n_\gamma \geq n > n_\beta$, if there are two public firms in the market, as under full integration these firms maximise the aggregated political benefit, the competition in the product market is excessively high. In this interval, if the two public firms are privatised, under full integration the competition in the product market is excessively low. Under partial integration, in this interval, as the supra-national authority privatises only one public firm and the other public firm maximises the political benefit of its country, the competition in the product market takes an intermediate level. Therefore, in this interval, a greater aggregated political benefit is obtained under partial integration than under full integration.
government by way of a monetary transference, and both countries would obtain a greater political benefit under partial integration than under full integration.

In the interval \( n \geq n > n_\delta \) under full integration both public firms are privatised while under partial integration only one public firm is privatised. It must be noted that if both public firms are privatised the same aggregated political benefit is obtained under full and partial integration. In this last type of integration, if there is one public firm in the market the aggregated political benefit is greater than if all firms are private \((W_{NP} + W_{PN} > 2W_{PP})\). However, if there is only one public firm in the market, the government which does not own the public firm obtains a lower political benefit than if all firms are private \((W_{PN} < W_{PP})\). But, as \( W_{NP} + W_{PN} > 2W_{PP} \), both countries would obtain a greater political benefit under partial integration than under full integration if the government which does not own the public firm is compensated by the other government by way of a monetary transference.

By using propositions 4 and 5, we can solve the first stage of the model when the consumer surplus has the same weight as the producer surplus in political profit. We obtain the following result.

**Proposition 6.** When political weights are equal, the decision whether to privatise the public firms is delegated to a supra-national authority, but the ownership of the public firms is held by governments.

Next we consider that the weights attached to the arguments of political benefit are different, in order to analyse whether the results obtained in the preceding sections are robust to changes in the parameter \( \alpha \).

### 6. Non-integration, partial integration and full integration when political weights differ

Using lemmas 1, 3 and 5 we obtain that, under non-integration, \( W_{NN} > W_{PN} \) if and only if \( n < n_{NI1} \), and \( W_{NP} > W_{PP} \) if and only if \( n < n_{NI2} \); \( n_{NI2} > n_{NI1} \) if and only if \( \alpha < 1.7142 \). The values of \( n_{NI1} \) and \( n_{NI2} \) are collected in the appendix and are shown in figure 1, in which we can identify four zones. Zone I groups the values of \( n \) such as \( n \leq \min\{n_{NI1}, n_{NI2}\} \). Zone II groups the values
of $n$ such as $n_{NI2} > n > n_{NI1}$. Zone III groups the values of $n$ such as $n > \max\{n_{NI1}, n_{NI2}\}$. Finally, zone IV groups the values of $n$ such as $n_{NI1} > n > n_{NI2}$.

**Proposition 7.** *Under non-integration, in zones I and IV neither government privatises its public firm, in zone II only one government privatises and in zone III both governments privatise.*

When we consider that $\alpha=1$ (see proposition 1) we obtain that if $n$ is high enough only one government privatises and if $n$ is low enough neither government privatises. In this case, as the consumer surplus has the same weight as the producer surplus in political benefit, competition in the product market is never high enough for both public firms to be privatised. However, when $\alpha$ is greater than 1, both governments privatise if parameter $n$ is high enough, for a given value of parameter $\alpha$ (zone III in figure 1). In this case, the weight of the consumer surplus in political benefit is sufficiently greater than that of the producer surplus and the competition in the product market is high enough for both public firms to be privatised.

In zone II in figure 1, only one government privatises. This zone exists only if $\alpha$ is low enough ($\alpha<1.7142$). As we show in proposition 1, the equilibrium obtained in this zone arises for strategic reasons. If $\alpha$ is high enough, as the weight of the consumer surplus is great enough, efficiency reasons dominate and, thus, either both governments privatise or neither does.

In zones I and IV of figure 1 neither government privatises its public firm since competition in the product market is low enough for all $\alpha$. In zone IV there are two equilibria: in one of them both governments privatise and in the other neither government privatises. It can be shown that the second equilibrium Pareto dominates the first.

Using lemmas 1, 3 and 5 we obtain that, under partial integration, $2W^{NN}>W^{NP}+W^{PN}$ if and only if $n<n_{PP1}$, $2W^{NN}>2W^{PP}$ if and only if $n<n_{PP2}$ and $W^{NP}+W^{PN} > 2W^{PP}$ if and only if $n<n_{PP3}$,
where \( n_{P1} < n_{P2} < n_{P3} \). The values of \( n_{Pli} \) \((i=1, 2, 3)\) are collected in the appendix. Using the preceding computations, the following result, illustrated in figure 2, is obtained.

**Proposition 8.** Under partial integration, if \( n \leq n_{P1} \) neither public firm is privatised (zone I'), if \( n_{P2} > n > n_{P1} \) the supra-national authority privatises only one (zone II') and if \( n > n_{P3} \) the supra-national authority privatises both of them (zone III').

[INSERT FIGURE 2 AROUND HERE]

The intuition underlying this result is the same as in proposition 2. For a given value of parameter \( \alpha \), when the number of private firms increases competition in the product market also increases; thus, if parameter \( n \) is low enough neither public firm is privatised, if parameter \( n \) takes an intermediate value only one is privatised and if parameter \( n \) is large enough both are privatised.

Using lemmas 5, 7 and 8 we obtain that, under full integration, \( 2W^{NN} > W^{NP} + W^{PN} \) if and only if \( n < n_{FI1} \), \( 2W^{NN} > 2W^{PP} \) if and only if \( n < n_{FI2} \), and \( W^{NP} + W^{PN} > 2W^{PP} \) if and only if \( n < n_{FI3} \), where \( n_{FI1} < n_{FI2} < n_{FI3} \). The values of \( n_{FIi} \) \((i=1, 2, 3)\) are collected in the appendix. Using the preceding computations, the following result, illustrated in figure 3, is obtained.

**Proposition 9.** Under full integration, if \( n \leq n_{FI2} \) neither of the public firms is privatised (zone I''); if \( n > n_{FI2} \) the supra-national authority privatises both of them (zone II'').

[INSERT FIGURE 3 AROUND HERE]

As we showed in proposition 3, given that the production costs are quadratic, it is cheaper to produce a given output level using two public firms than using one. Therefore, the supra-national authority never privatises only one public firm: it privatises either both or neither.
After analysing the three types of integration when political weights differ, we compare them in order to study whether the results obtained in section 5 are robust to changes in parameter $\alpha$. We obtain the following result.

**Proposition 10.** *When political weights differ, the aggregated political benefit under partial integration is no less than under full integration and non-integration.*

We saw in section 5 that when $\alpha=1$, the aggregated political benefit is greater or equal under partial integration than under non-integration. Political benefit is greater (equal) under partial integration when the equilibria obtained in the two cases are different (the same). This result is also obtained when $\alpha$ is greater than 1 since under partial integration there is a supra-national authority which decides, for reasons of efficiency, whether public firms are privatised. Under non-integration, the government of each country decides, for strategic reasons, whether its public firm is privatised or not.

Next we shall compare partial and full integration. When $\alpha=1$, the aggregated political benefit is no less under partial integration than under full integration. This result is also obtained when $\alpha$ is greater than 1.

When the parameters $\alpha$ and $n$ are such that the supra-national authority privatises both public firms, we obtain that the aggregated political benefit is the same under partial and full integration since private firms produce the same output level in both cases. When the political weights differ and the parameters $\alpha$ and $n$ are such that the supra-national authority does not privatisate either public firm, the aggregated political benefit is greater under partial integration than under full integration. When $\alpha$ is greater than 1, the output level of the public firms is greater under full integration than under partial integration, since public firms are much more aggressive in the first case than in the second. We saw in section 5 that when $\alpha=1$ the aggregated political benefit is the same under partial and full integration due to the symmetry of the model.

When the parameters $\alpha$ and $n$ are such that different equilibria are obtained under partial and full integration, we obtain that the aggregated political benefit is greater under partial integration than under full integration. This is because, given that the objective function of
public firms is aggregated political benefit, these firms are very aggressive in the product market and, thus, competition in the product market is very high.

7. Conclusions

The literature on mixed oligopoly does not analyse whether governments want to delegate the decision whether to privatise public firms in different countries to a supra-national authority. To analyse this idea, we consider three possible cases: non-integration, partial integration and full integration.

By comparing these three cases, when political weights are equal, we find that the aggregated political benefit is no less under partial integration than under full integration or non-integration. In a comparison between non-integration and partial integration we obtain that if the number of private firms is low enough no public firm is privatised in either case and the aggregated political benefit of the two countries is therefore the same in both cases. If the number of private firms is high enough, the aggregated political benefit is greater if countries integrate partially. This is due to the fact that under non-integration governments decide whether to privatised for strategic reasons, but under partial integration the supra-national authority decides whether or not to privatised for reasons of efficiency. As a result, the aggregated political benefit is greater in the latter case than in the former.

A comparison of partial and full integration cases shows that if the number of private firms is low (high) enough, neither public firm is (both are) privatised, so the aggregated political benefit obtained is the same in both cases. When there is an intermediate number of private firms, we obtain that the aggregated political benefit is greater under partial integration than under full integration, even though in the former case each public firm chooses the level of production that maximises the political benefit of its country, while in the latter it choose the level of production that maximises the aggregated political benefit.

When consumer surplus has a greater weight than producer surplus in political benefit, we also obtain that the aggregated political benefit is no less under partial integration than under non-integration. Under partial integration the aggregated political benefit is also no less than under full integration since, in this last case, public firms are excessively aggressive.
It should be noted that we have not considered asymmetries in the paper. For example, we could assume that there is a different number of private firms in each country or that the weight that the consumer surplus has in political benefit differs from one country to the other. These facts could modify the results obtained in the paper, but this issue is left for further research.

Appendix

Under non-integration, $W_{NN} = W_{PN}$ for those values of $\alpha$ and $n$ such that:

$$540-108n+252n^2+100n^3+144\alpha+273n\alpha-332n^2\alpha-100n^3\alpha+72\alpha^2+48n\alpha^2+60n^2\alpha^2-18\alpha^3-9n\alpha^3=0.$$ 

The value of the parameter $n$ that solves the preceding equation is:

$$n_{NI1} = \frac{1}{150(\alpha - 1)} \left( 126 - 166\alpha + 30\alpha^2 + A \frac{1}{B + 675(\alpha - 1)C} \right)^{1/3} + (B + 675(\alpha - 1)C)^{1/3},$$

where: $A = 23976 - 70407\alpha + 51991\alpha^2 + 225\alpha^3$, $B = 12643776 - 31118823\alpha + 27981558\alpha^2 - 13322701\alpha^3 + 4210065\alpha^4 - 391500\alpha^5 + 739638513\alpha^6 - 319702371\alpha^7 + 142006142\alpha^8 - 92511842\alpha^9 + 5294493\alpha^{10} + 224208\alpha^{11} + 7695\alpha^{12}$.

It can be shown that $W_{NN} > W_{PN}$ if and only if $n < n_{NI1}$.

Under non-integration, $W_{PN} = W_{PP}$ for those values of $\alpha$ and $n$ such that:

$$60+12n+40n^2+16n^3+62\alpha+51n\alpha-36n^2\alpha-16n^3\alpha-9\alpha^2-9n\alpha^2=0.$$ 

The value of the parameter $n$ that solves the preceding equation is:

$$n_{NI2} = \frac{1}{24(\alpha - 1)} \left( 20 - 18\alpha - 22\alpha^{1/3} \left( \frac{1}{E + 3\sqrt[3]{\alpha - 1}F} \right)^{1/3} + 2^{2/3}(E + 3\sqrt[3]{\alpha - 1}F)^{1/3} \right),$$

where $E = 320620032 - 819388224\alpha - 319702371\alpha^2 - 224208\alpha^3 + 7695\alpha^4 + 34307145\alpha^5 + 5294493\alpha^6 + 224208\alpha^7 + 7695\alpha^8$, $F = 320620032 - 819388224\alpha - 319702371\alpha^2 - 224208\alpha^3 + 7695\alpha^4 + 34307145\alpha^5 + 5294493\alpha^6 + 224208\alpha^7 + 7695\alpha^8$.
where: \( D = -64 + 297\alpha - 261\alpha^2 + 27\alpha^3 \), \( E = 7400 - 14202\alpha + 5535\alpha^2 + 1512\alpha^3 - 243\alpha^4 \), \( F = (1989312 - 3265504\alpha - 1000260\alpha^2 + 4611168\alpha^3 - 730458\alpha^4 - 76545\alpha^5 - 2916\alpha^6)^{1/2} \).

It can be shown that \( W^{NP} > W^{PP} \) if and only if \( n < n_{NI}^2 \), where \( n_{NI}^2 > n_{NI}^1 \) if and only if \( \alpha < 1.7142 \).

Under partial integration, \( 2W^{NN} = W^{NP} + W^{PN} \) for those values of \( \alpha \) and \( n \) such that:

\[
810 + 1278n + 84n^2 - 68n^3 - 1359\alpha - 1407n\alpha - 4n^2\alpha + 68n^3\alpha + 252\alpha^2 - 2n^2\alpha + 12n^2\alpha^2 - 9\alpha^3 - 9n\alpha^3 = 0.
\]

The value of the parameter \( n \) that solves the preceding equation is:

\[
n_{PI1} = \frac{1}{102(\alpha - 1)} \left( -2(21 - 3\alpha^2) + G \left( \frac{1}{H + 459(\alpha - 1)I} \right)^{1/3} + (H + 459(\alpha - 1)I)^{1/3} \right),
\]

where: \( G = 66942 - 137103\alpha + 82579\alpha^2 - 11757\alpha^3 + 495\alpha^4 \), \( H = -10500732 + 32078160\alpha - 34522272\alpha^2 + 14363225\alpha^3 - 1427823\alpha^4 - 34209\alpha^5 + 4347\alpha^6 \), \( I = (900495144 - 3749972868\alpha + 6461427294\alpha^2 - 5949900399\alpha^3 + 3174084583\alpha^4 - 1003842935\alpha^5 + 187896150\alpha^6 - 20420433\alpha^7 + 1238031\alpha^8 - 8637\alpha^9 + 486\alpha^{10})^{1/2} \).

It can be shown that \( 2W^{NN} > W^{NP} + W^{PN} \) if and only if \( n < n_{PI1} \).

Under partial integration, \( 2W^{NN} = 2W^{PP} \) for those values of \( \alpha \) and \( n \) such that:

\[
45 + 75n + 8n^2 - 72\alpha - 80n\alpha - 4n^2\alpha + 4n^3\alpha + 9\alpha^2 + 9n\alpha^2 = 0.
\]

The value of the parameter \( n \) that solves the preceding equation is:

\[
n_{PI2} = \frac{1}{6(\alpha - 1)} \left( 2(-2 + \alpha) + J \left( \frac{1}{H + 459(\alpha - 1)I} \right)^{1/3} + (K + 3\sqrt{3(\alpha - 1)L})^{1/3} \right),
\]

where: \( J = 241 - 481\alpha + 271\alpha^2 - 27\alpha^3 \), \( K = -2629 + 7935\alpha - 8391\alpha^2 + 3401\alpha^3 - 324\alpha^4 \),
\[ L = (-262440 + 1033959 \alpha^2 - 1353039 \alpha^3 - 604926 \alpha^4 + 143721 \alpha^5 - 16605 \alpha^6 + 729 \alpha^7)^{1/2}. \]

It can be shown that \( 2W^{NN} > 2W^{PP} \) if and only if \( n < n_{P12} \).

Under partial integration, \( W^{NP} + W^{PN} = 2W^{PP} \) for those values of \( \alpha \) and \( n \) such that:

\[ 90 + 158n + 24n^2 - 8n^3 - 137 \alpha - 161n\alpha - 16n^2 \alpha + 8n^3 \alpha + 9\alpha^2 + 9n\alpha^2 = 0. \]

The value of the parameter \( n \) that solves the preceding equation is:

\[ n_{P3} = \frac{1}{48(\alpha - 1)} \left( -48 + 32\alpha - 42^{2/3}M \left( \frac{1}{N + 3 \sqrt{6(\alpha - 1)P}} \right)^{1/3} + 2^{2/3}(M + 3 \sqrt{6(\alpha - 1)P})^{1/3} \right), \]

where:

\[ M = -546 + 1053\alpha - 542\alpha^2 + 27\alpha^3, \quad N = -14256 + 41760\alpha - 41958\alpha^2 + 15232\alpha^3 - 810\alpha^4, \]

\[ P = (-2264984 + 8300324\alpha - 11907098\alpha^2 + 8333931\alpha^3 - 2883420\alpha^4 + 448308\alpha^5 - 30294\alpha^6 + 729\alpha^7)^{1/2}. \]

It can be shown that \( W^{NP} + W^{PN} > 2W^{PP} \) if and only if \( n < n_{P3} \), where \( n_{P11} < n_{P2} < n_{P3} \).

Under full integration, \( 2W^{NN} > W^{NP} + W^{PN} \) if and only if \( n < n_{F11} \), \( 2W^{NN} > 2W^{PP} \) if and only if \( n < n_{F12} \), and \( W^{NP} + W^{PN} > 2W^{PP} \) if and only if \( n < n_{F3} \), where:

\[ n_{F11} = \frac{1}{4} (-1 + \sqrt{217 - 126\alpha + 18\alpha^2}), \quad n_{F12} = \frac{1}{2} (-1 + \sqrt{28 - 9\alpha}), \]

\[ n_{F13} = \frac{1}{4} (-3 + \sqrt{73 - 18\alpha}), \quad n_{F11} < n_{F2} < n_{F3}. \]
References


